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The gauge-invariant dynamic equations for current-carrying plasma-like media with topological defects

Nicholas B Volkov†

Institute of Electrophysics, Ural Division, Russian Academy of Sciences, 34 Komsomolskaya St, 620049 Ekaterinburg, Russia

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Abstract. A closed set of the gauge-invariant dynamic equations for a current-carrying plasma-like medium with dislocation-type and disclination-type topological defects together with the conditions at strong discontinuities is obtained using the variational principle and discussed. The dislocation and disclination fields, which compensate the non-homogeneity of the action of the gauge group $G = SO(3) \supset T(3)$, are described in the present theory by inexact external differential forms. The set of the Cartan structural equations for these forms has a direct correlation with the continuity equations for topological defects. The integrability conditions for the equations describing the dynamics of topological defects are obtained. It is shown that the integrability condition for the equation for disclination fields is equivalent to the balance equation for the angular momentum of the plasma-like medium together with the magnetic field. This condition is degenerated in the requirement of symmetry of the total stress tensor in the case of lack of topological defects. It is also shown that the total tensor of an energy–momentum of the plasma-like medium and of the magnetic field satisfies the balance equation.

1. Introduction

The aim of the present work is to obtain dynamic equations for current-carrying plasma-like media (we can classify with them, with minor reservations, unmagnetized solid and liquid metals), suitable for a theoretical research of non-equilibrium phase transitions. It is known [1, 2] that the passage of an electric current in a plasma-like medium is accompanied by excitation of various types of instabilities. These instabilities lead to non-equilibrium phase transitions, typical examples of which are the electrical explosion of conductors [3–8] and the structuring of an electric current at the cathode surface in vacuum arcs (formation of current cells [9, 10] or ectons, on nomenclature [11]). For the mentioned non-equilibrium phase transitions, the presence of a threshold value of the electric current I_* , and the generation of low-temperature plasma with a condensed disperse phase and of high-speed plasma jets is characteristic. We have shown [8, 12] that hot points or spots [13] are the source of plasma jets in an electrical explosion of a conductor. The hot points arise as a result of the localization of the Joule heat source due to the formation of large-scale vortex (hydrodynamic and current) structures in a current-carrying conductor. (The nonlinear coupling of these structures with perturbations of the conductor surface is also responsible for the stratification of the exploding conductor [14, 15].)

It is customary to assume [11] that if special measures are not taken to excite primary current cells, their nucleation happens as a result of an electrical explosion of micropoints

† E-mail address: nbv@ami.intec.ru

on the cathode surface owing to their heating by the emission current. The current cells are also observed on the surface of liquid–metal cathodes [9]. As an electric field promotes the growth of micropoints from a liquid metal [16, 17], it is supposed that in this case their explosion initiates the current cells as well. However, if we assume the possibility of the existence of mechanisms of localization of the Joule heat source in solid metal as offered in [12], the formation of current cells is possible on the ‘perfectly smooth’ surface of the cathode as well. There exists experimental evidence that the electrodynamic processes occurring in the surface layer of a cathode play the dominant role in the formation of current cells. (By comparing the results of experiments with film and massive cathodes [9] it has been established that since the width $h = 1.5 \times 10^{-4}$ cm the results of experiments on film and massive cathodes coincide.) In solid bodies, topological defects, such as dislocations and disclinations† are equivalents of large-scale structures. At the solid body boundary, either the Burgers vector is zero or the dislocation filament is perpendicular to the surface [19], much as vortex filaments in a liquid are bear up against the walls of the vessel or against its free surface, or they are closed [20]. The passage of the current across the dislocation filaments is hampered; therefore, the current has to ‘flow over’ the dislocation filaments and to approach the cathode surface in between the filaments. This will lead eventually to the formation of hot points in the surface layer of the cathode, which initiate primary current cells.

The physical phenomena above are characterized, first, by the threshold nature and, secondly, by the dominant role of topological defects as vortices in liquids, as dislocations and disclinations in solid bodies and of the physical processes on surfaces of the external and internal strong discontinuities. Thus, there is a need to have models of plasma-like media with topological defects for the explanation of these phenomena. (The present work is devoted to the construction of such a model.) Topological defects have the result that the equation of motion

$$x^i = F^i(X^\alpha, t) \quad (1)$$

(with X^α ($\alpha = 1, 2, 3$) being the coordinates of the reference configuration and x^i ($i = 1, 2, 3$) being the coordinates of the current (actual) configuration of the body (the three-dimensional domain occupied by the medium)) is not a one-to-one differentiable mapping (diffeomorphism). In the physics of real crystals [21] it is supposed that the disclinations and dislocations are the responses of the dynamical system to the break of the rotary and the translation symmetry, respectively. Therefore, it is possible to use the methods of the theory of the Yang–Mills fields [22] in the construction of continual models of the topological defects. (We shall mention papers [19, 23, 24], in which the tools of the gauge fields are used for solving various problems of the theory of condensed matter. In the first paper, the most full gauge-invariant theory of the dislocations and disclinations non-interacting with the electromagnetic field and quasi-particle excitations is constructed for isotropic solid bodies.)

In the present work we construct a model of the plasma-like medium with topological defects, using for their description a method offered in [19]. Thus we simultaneously obtain both the field equations and the boundary conditions, including the conditions at the surfaces

† Two types of topological defects (‘dislocations’) for sources of internal stresses had been entered into the continuum mechanics by Volterra [18] as far back as 1907. The first type of Volterra dislocations is connected to a break of the translation symmetry (it is the response to non-homogeneous action of the translation group $T(3)$); the second type is connected to a break of the rotary symmetry (it is the response to non-homogeneous action of the rotation group $SO(3)$). At the moment, the translation Volterra dislocations are known as dislocations, and the rotary Volterra dislocations are known as disclinations.

of strong discontinuities, using, as in [19], exact and inexact exterior differential forms (in the appendix we present the minimum of information about them necessary for our aims) and the base variational equation [25–27]

$$\delta \int L \tilde{\mu} + \delta \tilde{W}^* + \delta \tilde{W} = 0. \tag{2}$$

In equation (2) L is the Lagrangian, $\tilde{\mu}$ is the element of a four-dimensional volume (see the appendix), $\delta \tilde{W}^*$ is the functional determining the specified inflow of energy to the four-dimensional domain occupied by the medium, and $\delta \tilde{W}$ is the functional describing the additional inflow of energy due to the power interactions at the surface of the three-dimensional domain occupied by the medium. As shown in [27], equation (2) is the variational analogue of the energy balance equation. (For discontinuous functions, the first generalized derivatives of which are measures (belonging to the space BV) [28], the integration in equation (2) should be understood as integration with respect to the measure.)

2. The Lagrangian for a current-carrying plasma-like medium with topological defects

2.1. The Lagrangian for ‘vacuum’ states

2.1.1. The initial point of the construction of the Lagrangian for slowly varying (‘vacuum’, in relation to quasi-particle excitations) states of a plasma-like medium with topological defects is the Lagrangian for a continuous defectless medium, in particular, the Lagrangian for the elasticity theory. Therefore, we assume, according to [19], that there exist two spaces: the space of reference configurations E_3 (a three-dimensional Euclidean space with a global Cartesian coordinate covering $\{X^1, X^2, X^3\}$; the system of reference configurations of an elastic body represents a connected set Ω_3 of non-zero measures of the Euclidean volume, contained in a star-shaped domain Σ of the space E_3), and the space of current (actual) configurations *E_3 with a global coordinate covering $\{x^1, x^2, x^3\}$. The spacetime evolution of the elastic continuum is determined by the diffeomorphism

$$\mathbf{F} : \Omega_3 \times [t_0, t_1] \implies {}^*E_3 \times [t_0, t_1] |_{x^i = F^i(X^a, t)}$$

for which the action

$$\begin{aligned} \tilde{I}[\mathbf{F}] &= \int_{t_0}^{t_1} \int_{\Omega_3} L_0(X^a, F^i, \partial_a F^i) dX^1 \wedge dX^2 \wedge dX^3 \wedge dt \\ &= \int_{t_0}^{t_1} \int_{\Omega_3} L_0(X^a, \mathbf{F}, \partial_a \mathbf{F}) \mu \wedge dt \end{aligned}$$

has a stationary value for all diffeomorphisms satisfying the same Dirichlet conditions. The Lagrangian L_0 satisfies the invariance conditions relative to the similar action of the group $G_0 = SO(3)_0 \triangleright T(3)_0$ ($SO(3)_0$ and $T(3)_0$ are the orthogonal rotation group and the translation group, respectively) acting on the state vector \mathbf{F} by the rule

$${}^*\mathbf{F} = \mathbf{A}\mathbf{F} + \mathbf{b} \quad \mathbf{A} \in SO(3)_0 \quad \mathbf{b} \in T(3)_0$$

where \mathbf{A} is the orthogonal matrix of the constants and \mathbf{b} is the column vector of the constants.

According to the Noether theorem [29], the kinematics of the elastic continuum follows immediately from the statement of the variational principle and from the invariance conditions and leads to the nonlinear classical theory of an elastic continuum. On the other hand, the kinematics of an elastic continuum is uniquely determined by the diffeomorphism \mathbf{F} (equation (1)) and by the continuity and differentiability properties following from it [19].

(This remains valid for the discontinuous functions belonging to the space BV .) Actually, let us have $b^i = dF^i = \partial_a F^i dX^a$. It follows that $db^i = 0$ and $b^1 \wedge b^2 \wedge b^3|_{t=t^*} \neq 0$. On the other hand, from $db^i = 0$ and equation (A13) we obtain $b^i = d\hat{H}b^i$, whence it follows that $b^i = dF^i$, where $F^i = \hat{H}b^i + k^i$ (\hat{H} is the homotopy operator (see equation (A9) in the appendix); if $\{X^\alpha\}$ are the integral coordinates of ions measured in terms of the translation vectors of the lattice in the reference configuration $\{a_\alpha\}$, then we have $b^i = a_\alpha^i dX^\alpha$) and

$$b^1 \wedge b^2 \wedge b^3|_{t=t^*} = \frac{\partial(F^1, F^2, F^3)}{\partial(X^1, X^2, X^3)} \Big|_{t=t^*} \neq 0.$$

In the physics of a deformable continuum with topological defects, current configurations of a body are considered which cannot be obtained from the reference configuration with the help of the diffeomorphism F . According to [19], the simplest way for their introduction is the substitution of the conditions

$$b^i = dF^i = \partial_a F^i dX^a \quad db^i = 0 \quad b^1 \wedge b^2 \wedge b^3|_{t=t^*} \neq 0 \quad (3)$$

by the relations

$$\tilde{B}^i = \tilde{B}_a^i dX^a \quad d\tilde{B}^i \neq 0 \quad \tilde{B}^1 \wedge \tilde{B}^2 \wedge \tilde{B}^3|_{t=t^*} \neq 0 \quad (4)$$

($\{\tilde{B}^i\}$ is the 1-form of a distortion; $d = dX^a \wedge \partial_a$ is the four-dimensional external differentiation operator (see the appendix)). Applying the linear homotopy operator \hat{H} , introduced in equation (A9) and being a converse to the external differentiation operator on the submodule $\Xi(S)$ of all inexact forms defined on star-shaped domain S to equation (4), we obtain [19]

$$\tilde{B}^i = d\hat{H}\tilde{B}^i + \hat{H}d\tilde{B}^i = dF^i + \hat{H}d\tilde{B}^i \quad F^i = \hat{H}\tilde{B}^i + k^i. \quad (5)$$

According to equation (5), the independent 1-forms $\{\tilde{B}^i\}$ lead to the appearance of a completely integrable part, dF^i , and of an inexact (non-integrable) part, $\hat{H}d\tilde{B}^i$, defined by the 2-form $d\tilde{B}^i$. It is obvious that the part $\hat{H}d\tilde{B}^i$ is connected to the internal degrees of freedom (of symmetries) of the elastic continuum with topological defects. (It is just these degrees of freedom that prevent the realization of the diffeomorphism F , which determines in a unique manner the current configuration in the case of an elastic defectless body.) In the physics of real crystals [21] it is customary to assume that topological defects such as dislocations and disclinations are the response to the non-homogeneous action of the group $G = SO(3) \triangleright T(3)$, relative to which the Lagrangian L_0 is not invariant. Just the appearance of defects (of compensating fields [22]) restores the invariance of the original Lagrangian of the elasticity theory.

A theoretical description of this situation requires, according to the theory of Yang–Mills fields [22, 30], that the external differentiation operator $d = dX^a \wedge \partial_a$ be substituted by the external covariant differentiation operator (see the appendix) $D = dX^a \wedge D_a$ ($D_a = \partial_a + \tilde{G}_a$ is the covariant derivative; \tilde{G}_a is the 1-form of connectedness [31, 32]). As the group $G = SO(3) \triangleright T(3)$ is not semi-simple, it has no matrix representation acting on the vector F on the left. The authors of [19] were able to find such a representation and to specify the construction of the minimal substitution that is necessary for restoring the invariance of the original Lagrangian relative to the non-homogeneous action of the group $G = SO(3) \triangleright T(3)$ on the state vector F

$$*F = \mathbf{A}F + \mathbf{b} \quad \mathbf{A} \in SO(3) \quad \mathbf{b} \in T(3).$$

This minimal substitution has the form [19]

$$dF \implies \tilde{\mathbf{B}} = D\mathbf{F} + \mathbf{Y} = d\mathbf{F} + \tilde{\mathbf{G}}\mathbf{F} + \mathbf{Y}. \quad (6)$$

In equation (6) $\tilde{\mathbf{G}}$ is the (3×3) -matrix of the 1-forms of connectedness, taking in the Lie algebra of the group $SO(3)$ the values

$$\mathbf{G} = W^\alpha \mathbf{g}_\alpha \tag{7}$$

where \mathbf{g}_α are the infinitesimal generating (3×3) -matrices of the rotation group $SO(3)$ satisfying the known commutative relations [31–33]

$$[\mathbf{g}_\alpha, \mathbf{g}_\beta] = \mathbf{g}_\alpha \mathbf{g}_\beta - \mathbf{g}_\beta \mathbf{g}_\alpha = c_{\alpha\beta}^\varepsilon \mathbf{g}_\varepsilon. \tag{8}$$

$c_{\alpha\beta}^\varepsilon$ are the structural constants of the Lie algebra G satisfying the relations [31–33]

$$c_{\alpha\beta}^\varepsilon = -c_{\beta\alpha}^\varepsilon \tag{9}$$

and the Jacobi identities

$$c_{\alpha\beta}^\delta c_{\delta\gamma}^\varepsilon + c_{\beta\gamma}^\delta c_{\delta\alpha}^\varepsilon + c_{\gamma\alpha}^\delta c_{\delta\beta}^\varepsilon = 0. \tag{10}$$

Accordingly, in equation (6) \mathbf{Y} is the (3×3) -matrix of the 1-forms taking the values [19]

$$\mathbf{Y} = Y^i \mathbf{t}_i \tag{11}$$

$(\mathbf{t}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3})^T)$ are the infinitesimal generating matrices of the translation group $T(3)$. According to the consideration below, the 1-forms W^α and Y^i are the potentials of the disclination and dislocation fields, respectively.

Comparison of equation (5) with (6) shows that the 1-forms W^α and Y^i are inexact (they belong to the submodule of all inexact forms $\Xi(S)$) and, according to the property (A17) of the homotopy operator, they satisfy the inexact gauge conditions

$$X^a W_a^\alpha = 0 \quad X^a Y_a^i = 0 \quad (X_0^a = 0). \tag{12}$$

Matrices of the 1-forms $\tilde{\mathbf{B}}$, $\tilde{\mathbf{G}}$, and \mathbf{Y} are transformed according to the rules [19]

$$*\tilde{\mathbf{B}} = d*\mathbf{F} + *\tilde{\mathbf{G}}*\mathbf{F} + *\mathbf{Y} = \mathbf{A}\tilde{\mathbf{B}}. \tag{13}$$

$$*\tilde{\mathbf{G}} = \mathbf{A}\mathbf{G}\mathbf{A}^{-1} d\mathbf{A}\mathbf{A}^{-1}, \tag{14}$$

$$*\mathbf{Y} = \mathbf{A}\mathbf{Y} d\mathbf{b} - (\mathbf{A}\tilde{\mathbf{G}} d\mathbf{A})\mathbf{A}^{-1} \mathbf{b}. \tag{15}$$

(In equations (13)–(15) we have $\mathbf{A} \in SO(3)$, $\mathbf{b} \in T(3)$.)

To obtain the minimal substitution (6), the authors of [19] had to revise the description of the reference configuration accepted in the continuum mechanics [34–36]. (It is known that the reference configuration is necessary for the introduction of a quantitative measure for the relative and absolute strains as well as the Piola–Kirchhoff stress tensor defined relative to this configuration as a measure on surfaces [34].) As the reference configuration in the classical theory of elastic defectless continua, the so-called natural configuration in which the strains and stresses are lacking is used [34–36]. The authors of [19], having required the agreement of the dynamics of defects with the classical theory of elastic continua, defined the reference configuration as a configuration where strains, stresses, and defects are lacking (the requirement of the lack of defects is necessary to save the concept of deformation). In order to include correctly the collective quasi-particle excitations into the mechanics of the deformable continuum with defects we shall require that the reference configuration be characterized by a set of translation vectors $\{\mathbf{a}_\alpha, \alpha \in I_3\}$ and by the invariant metric tensor $\hat{g}_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$. The given metric tensor corresponds to a quite defined power spectrum of collective quasi-particle excitations (of conduction electrons, phonons, photons, etc), $\varepsilon_r(\hat{g}_{\alpha\beta}, \mathbf{p})$ (\mathbf{p} is the quasi-momentum of a quasi-particle excitation of the sort r), which, in the theory developed below, is assumed to be known. To save the requirements of the lack of strains and stresses, we shall also require that the external fields (for example, an electromagnetic or a temperature field) be uniform or lacking in the reference configuration.

We shall use as the coordinate covering of the space of reference configurations E_4 the Cartesian coordinate covering $\{X^\alpha, t\}$, where $\{X^\alpha\}$ is a set of integral coordinates of ions ('atoms') measured in the vectors of translation of the original lattice, $\{\mathbf{a}_\alpha\}$. Then the Cauchy strain tensor [19, 34–36]

$$\tilde{g}_{\alpha\beta} = \partial_\alpha F^i \delta_{ij} \partial_\beta F^j \quad (16)$$

coincides with the invariant metric tensor [21, 37] which, in the reference configuration, takes the form

$$\hat{g}_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta. \quad (17)$$

The relative strain tensor

$$\tilde{E}_{\alpha\beta} = \frac{1}{2}(\tilde{g}_{\alpha\beta} - \hat{g}_{\alpha\beta}) \quad (18)$$

is introduced to define the potential energy U of the deformable elastic continuum [34–36].

Below we shall restrict our consideration to the generally accepted square approximation [38] in the expression for the potential energy of the elastic continuum (here we do not specify the symmetry of the tensor of the elastic moduli, $M^{\alpha\beta\zeta\xi}$)

$$U = \frac{1}{2} M^{\alpha\beta\zeta\xi} E_{\alpha\beta} E_{\zeta\xi}. \quad (19)$$

The Lagrangian of the deformable elastic continuum is then equal to the difference between the kinetic energy

$$K = \frac{1}{2} \rho_0 \partial_4 F^i \delta_{ij} \partial_4 F^j$$

and the potential energy

$$\begin{aligned} L_0 = K - U(E_{\alpha\beta}) &= \frac{1}{2} \rho_0 \partial_4 F^i \delta_{ij} \partial_4 F^j - U(E_{\alpha\beta}) \\ &= \frac{1}{2} (\rho_0 \partial_4 F^i \delta_{ij} \partial_4 F^j - M^{\alpha\beta\zeta\xi} E_{\alpha\beta} E_{\zeta\xi}). \end{aligned} \quad (20)$$

In equation (20), ρ_0 is the mass density in the reference configuration. In the deformable continuum with topological defects, the construction of the minimal substitution in the form of (6) leads to the substitution of expressions (18) and (20) by relations

$$E_{\alpha\beta} = \frac{1}{2} (\tilde{B}_\alpha^i \delta_{ij} \tilde{B}_\alpha^j - \hat{g}_{\alpha\beta}) \quad (21)$$

$$L_0 = K - U(E_{\alpha\beta}) = \frac{1}{2} \rho_0 \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j - U(E_{\alpha\beta}) = \frac{1}{2} (\rho_0 \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j - M^{\alpha\beta\zeta\xi} E_{\alpha\beta} E_{\zeta\xi}). \quad (22)$$

The Lagrangian (22) is invariant relative to the non-homogeneous action of the gauge group $G = SO(3) \triangleright T(3)$. According to [22], the minimal substitution (6) requires that the Lagrangian

$$L_C = L_0 + \hat{s} \hat{L}$$

be introduced instead of $L_0(\tilde{B}^i)$, where \hat{L} is the Lagrangian of the compensating fields representing a function of the potentials of these fields and of their derivatives; \hat{s} is the coupling constant. (Obviously, the Lagrangian \hat{L} should be invariant relative to the non-homogeneous action of the gauge group $G = SO(3) \triangleright T(3)$.) Direct analogy with the theory of the Yang–Mills fields [22, 30] and with the Maxwell theory [39] allows us to obtain explicit expressions for \hat{L} . However, we first have to introduce, according to [19], definitions of the 3-forms of disclinations \tilde{O}^i

$$\tilde{O}^i = J_w^i \wedge dt + N_w^i = J_w^{\alpha i} \tilde{\mu}_\alpha \wedge dt + n_w^i \mu \quad (23)$$

and of the 2-forms of dislocations \tilde{D}^i

$$\tilde{D}^i = J_\varphi^i \wedge dt + n_\varphi^i = J_{\varphi, \alpha}^i dX^\alpha \wedge dt + n_\varphi^{\alpha i} \tilde{\mu}_\alpha. \quad (24)$$

It has been shown [19] that the phenomenological kinematic equations for topological defects like dislocations and disclinations are satisfied if only if the equations for exterior differential forms

$$d\tilde{O}^i = 0 \quad d\tilde{D}^i = \tilde{O}^i \quad (25)$$

are satisfied on the domain E_4 where spacetime evolution of the continuous medium occurs. In equations (23) and (24) (see the appendix, where the basis vectors of the spaces of exterior forms used are introduced),

$$N_\varphi^i = n_\varphi^{\alpha i} \tilde{\mu}_\alpha \quad (26)$$

are the 2-forms of the dislocation density;

$$J_\varphi^i = J_{\varphi,\alpha}^i dX^\alpha \quad (27)$$

are the 1-forms of the dislocation flux;

$$N_w^i = n_w^i \mu \quad (28)$$

are the 3-forms of the disclination density;

$$J_w^i = J_w^{\alpha i} \tilde{\mu}_\alpha \quad (29)$$

are the 2-forms of the disclination flux.

The continuity equations (25) permit a set of the first integrals [19]

$$\tilde{D}^i = d\tilde{B}^i + \tilde{K}^i. \quad (30)$$

In equation (30), the first term represents an external differential from the 1-form of the velocity distortion [19]

$$\tilde{B}^i = \tilde{V}^i dt + \tilde{b}_\alpha^i dX^\alpha = \tilde{B}_\alpha^i dX^\alpha. \quad (31)$$

The second term in equation (30) is expressed through the 2-form of the spin torsion as [19]

$$\tilde{K}^i = \tilde{w}^i \wedge dt + \tilde{k}^i = -\tilde{w}_\alpha^i dX^\alpha \wedge dt + \tilde{k}^{\alpha i} \tilde{\mu}_\alpha. \quad (32)$$

In equations (30) and (32),

$$\tilde{k}^i = \tilde{k}^{\alpha i} \tilde{\mu}_\alpha \quad (33)$$

are the 2-forms of the bend torsion;

$$\tilde{w}^i = \tilde{w}_\alpha^i dX^\alpha \quad (34)$$

are the 1-forms of the spin;

$$\tilde{b}^i = \tilde{b}_\alpha^i dX^\alpha \quad (35)$$

are the 1-forms of the distortion; \tilde{V}^i are the 0-forms of the velocity. In matrix form, equations (25) and (30) look like

$$d\tilde{O} = 0 \quad \tilde{O} = d\tilde{D} = d\tilde{K} \quad (36)$$

$$\tilde{D} = d\tilde{B} + \tilde{K}. \quad (37)$$

In equations (36) and (37) $\tilde{B} \in \Lambda_{3,1}^1(E_4)$ is the column vector whose components represent the 1-forms of the distortion $\{\tilde{B}^i\}$; $\tilde{K}, \tilde{D} \in \Lambda_{3,1}^2(E_4)$ are the column vectors with components being the 2-forms $\{\tilde{K}^i\}, \{\tilde{D}^i\}$; $\tilde{O} \in \Lambda_{3,1}^3(E_4)$ is the column vector of the 3-forms $\{\tilde{O}^i\}$.

The use of the construction of the minimal substitution in the form (6), of the inexact gauge, and of the set of Cartan structural equations representing a complete system of

differentials of exterior forms [32,40,41], and the identification of the matrix of the 2-forms of dislocations, $\tilde{\mathbf{D}}$, with the Cartan torsion, $\tilde{\mathbf{S}}$, and of the matrix of the 1-forms of distortion, $\tilde{\mathbf{B}}$, with the matrix of so-called adjoint 1-forms, $\tilde{\mathbf{h}}$ [32], has allowed the authors of [19] to obtain a representation of the phenomenologically introduced characteristics of defects $\tilde{\mathbf{B}}$, $\tilde{\mathbf{D}}$, $\tilde{\mathbf{K}}$, and $\tilde{\mathbf{O}}$ through compensating fields $\tilde{\mathbf{G}}$ and \mathbf{Y}

$$\tilde{\mathbf{B}} = d\mathbf{F} + \tilde{\mathbf{G}}\mathbf{F} + \mathbf{Y} = D\mathbf{F} + \mathbf{Y} \quad (38)$$

$$\tilde{\mathbf{D}} = \tilde{\mathbf{T}}\mathbf{F} + D\mathbf{Y} \quad (39)$$

$$\tilde{\mathbf{K}} = \tilde{\mathbf{G}} \wedge (D\mathbf{F} + \mathbf{Y}) = \tilde{\mathbf{G}} \wedge \tilde{\mathbf{B}} \quad (40)$$

$$\tilde{\mathbf{O}} = d(\tilde{\mathbf{T}}\mathbf{F} + D\mathbf{Y}) = d\tilde{\mathbf{D}}. \quad (41)$$

In equations (39) and (41),

$$\tilde{\mathbf{T}} = d\tilde{\mathbf{G}} + \tilde{\mathbf{G}} \wedge \tilde{\mathbf{G}} \quad (42)$$

is the matrix of the 2-forms of curvature. According to [19], matrices $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{T}}$ are transformed by the rules

$$*\tilde{\mathbf{D}} = \mathbf{A}\tilde{\mathbf{D}} \quad *\tilde{\mathbf{T}} = \mathbf{A}\tilde{\mathbf{T}}\mathbf{A}^{-1}. \quad (43)$$

Equations (38)–(41), written through the components of the corresponding matrices of exterior forms, look like

$$\tilde{B}^i = dF^i + \tilde{G}_j^i F^j + Y^i = dF^i + W^\alpha g_{\alpha j}^i F^j + Y^i \quad (44)$$

$$\tilde{D}^i = \tilde{T}_j^i F^j + DY^i = \tilde{C}^\alpha g_{\alpha j}^i F^j + DY^i = (dW^\alpha + \frac{1}{2}c_{\beta\gamma}^\alpha W^\beta \wedge W^\gamma)g_{\alpha j}^i F^j + DY^i \quad (45)$$

$$\tilde{K}^i = \tilde{G}_j^i \wedge (DF^j + Y^j) = W^\alpha g_{\alpha j}^i \wedge (DF^j + Y^j) \quad (46)$$

$$\tilde{O}^i = d\tilde{D}^i. \quad (47)$$

The so introduced definitions of the field characteristics of defects and the direct analogy to the Yang–Mills fields allow us to write the gauge-invariant Lagrangians for the disclination (L_W) and the dislocation (L_Y) fields as [19]

$$L_W = -\frac{1}{2}s_w c_{\alpha\beta} \tilde{C}_{ab}^\alpha g_w^{ac} g_w^{be} \tilde{C}_{ce}^\beta \quad (48)$$

$$L_Y = -\frac{1}{2}s_Y \delta_{ij} \tilde{D}_{ab}^i g_Y^{ac} g_Y^{be} \tilde{D}_{ce}^j. \quad (49)$$

In equation (48), $c_{\alpha\beta} = c_{\alpha\gamma}^\delta c_{\beta\delta}^\gamma = c_{\beta\alpha}$ is the Killing metric of the semi-simple group $SO(3)$ that defines the components of a non-singular matrix [31–33];

$$g_w^{\alpha\beta} = -\delta^{\alpha\beta} \quad g_w^{44} = \frac{1}{c_w^2} \quad g_w^{ab} = 0 \text{ at } a \neq b \quad (50)$$

$$\tilde{\mathbf{T}} = \tilde{C}^\alpha \mathbf{g}_\alpha \quad \tilde{C}^\alpha = \frac{1}{2}\tilde{C}_{ab}^\alpha dX^a \wedge dX^b \quad \tilde{C}_{ab}^\alpha = \partial_a W_b^\alpha - \partial_b W_a^\alpha + c_{\beta\gamma}^\alpha W_a^\beta W_b^\gamma. \quad (51)$$

\tilde{C}_{ab}^α is the tensor of the disclination fields, and s_w is the coupling constant. In equation (49),

$$\tilde{D}_{ab}^i = \partial_a Y_b^i - \partial_b Y_a^i + g_{\xi j}^i (W_a^\xi Y_b^j - W_b^\xi Y_a^j + \tilde{C}_{ab}^\xi F^j) \quad (52)$$

is the tensor of the dislocation fields,

$$g_Y^{\alpha\beta} = -\delta^{\alpha\beta} \quad g_Y^{44} = \frac{1}{c_Y^2} \quad g_Y^{ab} = 0 \text{ at } a \neq b. \quad (53)$$

s_Y is the coupling constant. In equations (50) and (53), $c_w > 0$ and $c_Y > 0$ are the propagation constants having dimensions of velocity. (They are equal to the velocity of light in vacuum c_0 only in the relativistic theory.)

Thus, the Lagrangian of the deformable continuum with topological defects, describing the slowly varying (vacuum, relative to the quasi-particle excitations) states has the form

$$L_C = L_0 + L_W + L_Y = \frac{1}{2} \rho_0 \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j - \frac{1}{2} M^{\alpha\beta\zeta\xi} E_{\alpha\beta} E_{\zeta\xi} - \frac{1}{2} s_w c_{\alpha\beta} \tilde{C}_{ab}^{\alpha c} g_w^{ac} g_w^{be} \tilde{C}_{ce}^{\beta} - \frac{1}{2} s_Y \delta_{ij} \tilde{D}_{ab}^i g_Y^{ac} g_Y^{be} \tilde{D}_{ce}^j. \quad (54)$$

In the Lagrangian (54), the mass density ρ_0 and the tensor of the elastic moduli $M^{\alpha\beta\zeta\xi}$ are defined in the reference configuration.

2.2. The Lagrangian for quasi-particle excitations

In constructing the Lagrangian for the quasi-particle excitations, we restrict ourselves to the consideration of conduction electrons and phonons, assuming that their power spectra for the reference configuration are known. Therefore, the initial point is the description of the quasi-particle excitations in a deformable defectless lattice.

2.2.1. We shall assume that $\{X^\alpha\}$ ($\alpha \in I_3$) are integral coordinates of ions measured in terms of the translation vectors of the lattice in the reference configuration $\{\mathbf{a}_\alpha\}$. The translation vectors define the invariant metric tensor $\hat{g}_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ of the original lattice. The physically infinitesimal (large in comparison with the period of the lattice, but small in comparison with the distances on which its characteristics change essentially [37]) differential for a motionless defectless lattice $d\mathbf{r}$ is defined as

$$d\mathbf{r} = d\mathbf{F} = \mathbf{a}_\alpha dX^\alpha. \quad (55)$$

(It can readily be seen that equation (55) is equivalent to (3).)

It is known [42] that the semiclassical wavefunction of an electron belonging to a certain power zone of a motionless lattice is determined by the asymptotics

$$\psi(X^\alpha, t) \sim \exp\left(\frac{i}{\hbar} S_0(X^\alpha, t)\right)$$

where S_0 is the semiclassical action. For a periodic motionless lattice, the Hamiltonian of the conduction electrons coincides with an energy $\varepsilon_e = \varepsilon_e(k_\alpha, \hat{g}_{\alpha\beta})$ which is a periodic function of the components of the invariant quasi-momentum k_α with a period of $2\pi\hbar$ and depends on the invariant parameters of a unit cell, defined for a defectless lattice by the tensor $\hat{g}_{\alpha\beta}$ or its converse $\hat{g}^{\alpha\beta}$. The derivatives of the semiclassical action determine the energy and the quasi-momentum of the conduction electrons in a motionless defectless lattice

$$\left(\frac{\partial S_0}{\partial t}\right)_{X^\alpha} = \varepsilon_e(k_\alpha, \hat{g}_{\alpha\beta}) \quad \left(\frac{\partial S_0}{\partial X^\alpha}\right)_t = k_\alpha.$$

The semiclassical action S in a deformable defectless lattice is defined by the transformation rules for one-electron wavefunctions in the Galilei conversions [43]:

$$S = S_0 + m\mathbf{v}\mathbf{r} - \frac{mv^2 t}{2} \quad (56)$$

where m is the mass of a free electron. Differentiating equation (56), it is possible to obtain an expression for the momentum through which the invariant quasi-momentum of the conduction electrons in a deformable lattice is expressed (we take this expression as its definition):

$$k_\alpha = \mathbf{a}_\alpha \cdot (\mathbf{p} - m\mathbf{v}). \quad (57)$$

The Hamiltonian for the conduction electrons in a deformable defectless lattice is defined by the derivative

$$\check{H}(\mathbf{p}, \mathbf{r}, t) = \left(\frac{\partial S}{\partial t} \right)_r = \varepsilon_e + \mathbf{p}\mathbf{v} - \frac{m\mathbf{v}^2}{2}. \quad (58)$$

In the Hamiltonian (58) $\varepsilon_e = \varepsilon_e(\mathbf{a}_\alpha(\mathbf{p} - m\mathbf{v}), \hat{g}_{\alpha\beta})$ is a periodic function of the quasi-momentum with a period of $2\pi\hbar$, which is defined by local values of the translation vectors of the direct (or inverse) lattice in a defectless material.

The electron energy $\tilde{\varepsilon}_e$ in a deformable lattice is determined by the Galilei conversions [35, 41] $\tilde{\varepsilon}_e = \varepsilon_e + \mathbf{v} \cdot \mathbf{p}_0 + m\mathbf{v}^2/2$. In the given equality, \mathbf{p}_0 is a mean value of the electron momentum in the reference system with $\mathbf{v} = 0$: $\mathbf{p}_0 = m\partial\varepsilon_e/\partial\mathbf{p}$. Hence, we have

$$\tilde{\varepsilon}_e = \varepsilon_e + m\mathbf{v} \frac{\partial\varepsilon_e}{\partial\mathbf{p}} + \frac{m\mathbf{v}^2}{2} = K_e + U_e \quad (59)$$

where, as distinct from the Hamiltonian (58), the energy $\tilde{\varepsilon}_e$ is a periodic function of the quasi-momentum. In equation (59),

$$K_e = m\mathbf{v} \frac{\partial\varepsilon_e}{\partial\mathbf{p}} + \frac{m\mathbf{v}^2}{2} \quad (60)$$

is the ‘kinetic’ energy of the electron,

$$U_e = \tilde{\varepsilon}_e \quad (61)$$

is the ‘potential’ energy of the electron.

The kinetic equation for conduction electrons interacting with an electromagnetic field in a defectless body has the form [37] (below $f_e(\mathbf{p}, \mathbf{r}, t)$ is the electronic cumulative distribution function in a current configuration)

$$\frac{\partial f_e}{\partial t} + \frac{\partial f_e}{\partial \mathbf{r}} \cdot \frac{\partial \check{H}}{\partial \mathbf{p}} - \frac{\partial f_e}{\partial \mathbf{p}} \cdot \frac{\partial \check{H}}{\partial \mathbf{r}} - e \left(\mathbf{E} + \frac{1}{c_0} \left(\frac{\partial \check{H}}{\partial \mathbf{p}} \times \mathbf{H} \right) \right) \frac{\partial f_e}{\partial \mathbf{p}} = \frac{\delta f_e}{\delta t}$$

where $\delta f_e/\delta t$ is the operator of collisions (below we do not define its form); \mathbf{E} and \mathbf{H} are the strengths of the electric and the magnetic field, respectively. The electronic cumulative distribution function is periodic:

$$f_e(\mathbf{p}, \mathbf{r}, t) = f_e(\mathbf{p} + 2\pi\hbar\mathbf{a}^\alpha(\mathbf{r}, t), \mathbf{r}, t).$$

In [37] it is shown that the acyclic functions can be eliminated from the kinetic equation, substituting the momentum \mathbf{p} by the invariant quasi-momentum k_α and considering the function $f_e(k_\alpha, \mathbf{r}, t)$:

$$\begin{aligned} \frac{\partial f_e}{\partial t} + \mathbf{v} \cdot \frac{\partial f_e}{\partial \mathbf{r}} + \mathbf{a}_\alpha \cdot \frac{\partial f_e}{\partial \mathbf{r}} \frac{\partial \varepsilon_e}{\partial k_\alpha} - \frac{\partial f_e}{\partial k_\alpha} \mathbf{a}_\alpha \cdot \frac{\partial \varepsilon_e}{\partial \mathbf{r}} \\ - e \left(\mathbf{a}_\alpha \cdot \mathbf{E}' + \frac{1}{c_0} \frac{\partial \varepsilon_e}{\partial k_\beta} \mathbf{H}' \cdot (\mathbf{a}_\alpha \times \mathbf{a}_\beta) \right) \frac{\partial f_e}{\partial k_\alpha} = \frac{\delta f_e}{\delta t}. \end{aligned} \quad (62)$$

In equation (62), we have denoted, as in [37],

$$\mathbf{H}' = \mathbf{H} - \frac{m c_0}{e} (\nabla \times \mathbf{v}) \quad \mathbf{E}' = \mathbf{E} + \frac{1}{c_0} (\mathbf{v} \times \mathbf{H}) + \frac{m}{e} \frac{d\mathbf{v}}{dt} \left(\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \right). \quad (63)$$

The kinetic equation (62) can be rewritten relative to the reference configuration (we note that we still consider a defectless body):

$$\frac{d\hat{f}_e}{dt} + \{\hat{f}_e, \varepsilon_e\} - e \left(\tilde{g}_{\alpha\gamma} E'^{\gamma} + \frac{1}{c_0} e_{ijl} \tilde{g}_{\alpha\beta}^{\prime j l} a_\gamma^i H'^{\gamma} \frac{\partial \varepsilon_e}{\partial k_\beta} \right) \frac{\partial \hat{f}_e}{\partial k_\alpha} = \frac{\delta \hat{f}_e}{\delta t}. \quad (64)$$

In equation (64), $\tilde{g}_{\alpha\beta}^{ij} = a_\alpha^i a_\beta^j$, $\tilde{g}_{\alpha\beta} = \delta_{ij} \tilde{g}_{\alpha\beta}^{ij}$. $E^i = a_\gamma^i E'^\gamma$, $H^i = a_\gamma^i H'^\gamma$ are the components of strengths of the electric and the magnetic field in a current configuration, respectively, E'^γ , H'^γ are the same in the reference configuration,

$$\{\hat{f}_e, \varepsilon_e\} = \delta_\beta^\alpha \left(\frac{\partial \hat{f}_e}{\partial X^\alpha} \frac{\partial \varepsilon_e}{\partial k_\beta} - \frac{\partial \hat{f}_e}{\partial k_\beta} \frac{\partial \varepsilon_e}{\partial X^\alpha} \right)$$

is the Poisson bracket. In equation (64), the electronic cumulative distribution function $\hat{f}_e(k_\alpha, X^\alpha, t)$ and the operator of collisions $\delta \hat{f}_e / \delta t$ are defined relative to the reference configuration.

We shall define the element of volume in the momentum space by the expression

$$\mu_k = dk_1 \wedge dk_2 \wedge dk_3 = \frac{1}{3!} e^{\alpha\beta\gamma} dk_\alpha \wedge dk_\beta \wedge dk_\gamma. \quad (65)$$

Then the normalization of the electronic cumulative distribution function in the reference configuration takes the form

$$\hat{n}_e = \langle \hat{f}_e \rangle = \int 2 \hat{f}_e \frac{\mu_k}{(2\pi\hbar)^3}. \quad (66)$$

As the initial point for the description of the conduction electrons in a plasma-like medium with defects of dislocation and disclination types, we shall use (57), (59), and (64). We offer for this purpose the following construction of the minimal substitution ($a^i = a_\alpha^i dX^\alpha$ are the exact 1-forms of the translation vectors)

$$dF^i \equiv a^i \implies \tilde{B}^i = dF^i + \tilde{G}_j^i F^j + Y^i \quad \hat{g}_{\alpha\beta} \implies \tilde{g}_{\alpha\beta} = \tilde{B}_\alpha^i \delta_{ij} \tilde{B}_\beta^j \quad (67)$$

$$\varepsilon_e = \varepsilon_e(k_\alpha, \hat{g}_{\alpha\beta}) \implies \varepsilon_e = \varepsilon_e(k_\alpha, \tilde{g}_{\alpha\beta}) \quad (68)$$

$$\tilde{\varepsilon}_e = \varepsilon_e + m \mathbf{v} \cdot \frac{\partial \varepsilon_e}{\partial \mathbf{p}} + \frac{m \mathbf{v}^2}{2} \implies \tilde{\varepsilon}_e = \varepsilon_e + m \delta_{ij} \tilde{B}_4^j \frac{\partial \varepsilon_e}{\partial p_i} + \frac{1}{2} m \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j. \quad (69)$$

The construction of the minimal substitution (67)–(69) leads to the following change of the kinetic equation for the electronic cumulative distribution function $\hat{f}_e(k_\alpha, X^\alpha, t)$:

$$\frac{d \hat{f}_e}{dt} + \{\hat{f}_e, \varepsilon_e\} - e \left(\tilde{g}_{\alpha\gamma} E'^\gamma + \frac{1}{c_0} e_{ijl} \tilde{g}_{\alpha\beta}^{jl} \tilde{B}_\gamma^i H'^\gamma \frac{\partial \varepsilon_e}{\partial k_\beta} \right) \frac{\partial \hat{f}_e}{\partial k_\alpha} = \frac{\delta \hat{f}_e}{\delta t}. \quad (70)$$

In equation (70), we shall designate $\tilde{g}_{\alpha\beta}^{ij} = \tilde{B}_\alpha^i \tilde{B}_\beta^j$, $\tilde{g}_{\alpha\beta} = \delta_{ij} \tilde{g}_{\alpha\beta}^{ij}$, $E^i = \tilde{B}_\gamma^i E'^\gamma$, $H^i = \tilde{B}_\gamma^i H'^\gamma$. Relation (69), in view of relations (60) and (61), allows us to obtain the Lagrangian for the conduction electrons in a plasma-like medium with topological defects, identifying it with the difference of their ‘kinetic’ and ‘potential’ energies averaged by the cumulative distribution function:

$$L_e = m \delta_{ij} \tilde{B}_4^j \left\langle \frac{\partial \varepsilon_e}{\partial p_i} \hat{f}_e \right\rangle + \frac{1}{2} m \langle \hat{f}_e \rangle \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j - \langle \varepsilon_e \hat{f}_e \rangle. \quad (71)$$

The Lagrangian (71) does not contain the contribution from the operator of electron–phonon collisions, as it is compensated by the similar contribution from phonon–electron collisions. It can readily be seen that the Lagrangian (71) is invariant relative to the non-homogeneous action of the gauge group $G = SO(3) \triangleright T(3)$.

2.2.2. Considering the contribution of phonons to the Lagrangian of a continuous plasma-like medium with topological defects, we assume that the slow motions (deformations) of the continuous medium can be separated from the fast motions (oscillations) of atoms in the deformed lattice with defects.

The energy of the oscillations of the lattice is known to look like [21, 44]

$$\varepsilon_{\text{ph}} = \varepsilon_{\text{ph},0} + \sum_{\mathbf{q}} \hbar\omega(\mathbf{q})N_{\mathbf{q}}. \quad (72)$$

In equation (72)

$$\varepsilon_{\text{ph},0} = \frac{\hbar}{2} \int_0^{\omega_m} \omega v(\omega) d\omega$$

is the energy of the zero oscillations of the lattice and $v(\omega)$ is the density of the phonon states. Relation (72) points to the fact that a weakly excited state of an ideal crystal is equivalent to the ideal gas of quasi-particles (phonons) the energy of each of which is equal to $\hbar\omega(\mathbf{q})$ and their number in each state is defined by a set of representation particle numbers $\{N_{\mathbf{q}}\}$ (\mathbf{q} is a quasi-wavevector). According to the de Broglie principle, the motion of each quasi-particle is characterized by the velocity $\mathbf{v} = \partial\varepsilon_{\text{ph}}/\partial\mathbf{p} = \partial\omega/\partial\mathbf{q}$ and by the quasi-momentum $\mathbf{p} = \hbar\mathbf{q}$.

The macroscopic state of a defectless crystal is determined by the mean representation particle number $f_s(\mathbf{q})$, that is the cumulative distribution function of phonons in the state s . For an equilibrium thermodynamic state of a defectless crystal the mean representation particle numbers are defined by the Bose cumulative distribution function [44]

$$\langle\langle N_s(\mathbf{q}) \rangle\rangle = f_0(\omega_s(\mathbf{q})) = \left(\exp\left(\frac{\hbar\omega_s}{k_{\text{B}}T_{\text{ph}}}\right) - 1 \right)^{-1}$$

where k_{B} is Boltzmann's constant, T_{ph} is the temperature of the phonon gas, $\langle\langle \dots \rangle\rangle$ denotes an average by the equilibrium thermodynamic state, including the quantum mechanical average and the statistical average over the Gibbs ensemble.

For $\delta L \gg \bar{\lambda}$ (δL is the distance on which the macroscopic characteristics of the crystal vary, $\bar{\lambda}$ is the mean wavelength of a phonon) the representations for phonons can also be saved for a deformable crystal, but, instead of individual normal coordinates, it is necessary to consider wavepackets belonging to an interval of wavevectors δq [21]:

$$\delta L \geq \frac{1}{\delta q} \geq \bar{\lambda} \quad \text{or} \quad \frac{\delta q}{q} \ll 1. \quad (73)$$

The wavepacket (73) can be put in correspondence with an oscillation with a quasi-wavevector \mathbf{q} , that is a phonon in the state \mathbf{q} , the velocity of which is defined by the group velocity of the wavepacket

$$\mathbf{v} = \frac{\partial\omega}{\partial\mathbf{q}} = \frac{\partial\varepsilon_{\text{ph}}}{\partial\mathbf{p}}.$$

If the space position of a phonon in a deformed lattice is measured accurate to δx ($\bar{\lambda} \ll \delta x \ll \delta L$) (condition $\delta q \delta x \sim 1$ does not contradict with inequalities (73) [21]), it is possible to assign a coordinate \mathbf{r} to the phonon whose quasi-wavevector is \mathbf{q} . In this case the inhomogeneous state of the crystalline lattice also reveals itself in the dependence of the frequency of the phonons of the sort s (of their Hamiltonian) on \mathbf{r}

$$\omega_s = \omega_s(\mathbf{p}, \hat{g}_{\alpha\beta}(\mathbf{r})) = \omega_s(\mathbf{q}, \hat{g}_{\alpha\beta}(\mathbf{r})). \quad (74)$$

Phonons have a Bose distribution only in the thermodynamic equilibrium. In the general case it is necessary to solve the kinetic equation for the cumulative distribution function of the phonons of the sort s : $f_s(\mathbf{q}, \mathbf{r}, t) = f_s(\mathbf{p}, \mathbf{r}, t)$. The number of phonons of the sort s in an element of volume of the phase space $\mu_q \mu$ is defined by the relation (μ_q is the element of volume in the momentum space (see (65))):

$$f_s(\mathbf{q}, \mathbf{r}, t) \frac{\mu_q \mu}{(2\pi\hbar)^3}.$$

Accordingly, the number of phonons of the sort s moving along some trajectory $\{\mathbf{r}(t), \mathbf{q}(t)\}$ is determined by the kinetic equation similar to the kinetic equation for the conduction electrons:

$$\frac{\partial f_s}{\partial t} + \mathbf{a}_\beta \cdot \frac{\partial f_s}{\partial \mathbf{r}} \frac{\partial \varepsilon_{\text{ph}}}{\partial \mathbf{q}_\beta} - \frac{\partial \varepsilon_{\text{ph}}}{\partial \mathbf{r}} \cdot \mathbf{a}_\beta \frac{\partial f_s}{\partial \mathbf{q}_\beta} = \frac{\delta f_s}{\delta t}$$

where $\{\mathbf{a}_\beta\}$ are the translation vectors, $\{q_\beta\}$ are the components of the invariant quasi-wavevector of a phonon, and $\delta f_s / \delta t$ is the operator of collisions.

The kinetic equation relative to the reference configuration takes the form

$$\frac{d \hat{f}_s}{dt} + \{\hat{f}_s, \varepsilon_{\text{ph}}\} = \frac{\delta \hat{f}_s}{\delta t}. \quad (75)$$

In equation (75),

$$\{\hat{f}_s, \varepsilon_{\text{ph}}\} = \delta_\beta^\alpha \left(\frac{\partial \hat{f}_s}{\partial X^\alpha} \frac{\partial \varepsilon_{\text{ph}}}{\partial q_\beta} - \frac{\partial \hat{f}_s}{\partial q_\beta} \frac{\partial \varepsilon_{\text{ph}}}{\partial X^\alpha} \right)$$

is the Poisson bracket, and $\hat{f}_s(q_\alpha, X^\alpha, t)$ and $\delta \hat{f}_s / \delta t$ are defined relative to the reference configuration.

As the initial point for the description of the phonons in a plasma-like medium with topological defects of the dislocation and disclination types, we shall use equations (74) and (75), offering the construction of a minimal substitution as follows:

$$dF^i \equiv a^i \implies \tilde{B}^i = dF^i + \tilde{G}_j^i F^j + Y^i \quad \hat{g}_{\alpha\beta} \implies \tilde{g}_{\alpha\beta} = \tilde{B}_\alpha^i \delta_{ij} \tilde{B}_\beta^j \quad (76)$$

$$\varepsilon_{\text{ph}} = \varepsilon_{\text{ph}}(q_\alpha, \hat{g}_{\alpha\beta}) \implies \varepsilon_{\text{ph}} = \varepsilon_{\text{ph}}(q_\alpha, \tilde{g}_{\alpha\beta}). \quad (77)$$

(For this substitution the kinetic equation (75) for the cumulative distribution function of phonons is not changed.)

Relation (77) allows us to obtain the Lagrangian for the phonons in the defective material, identifying it with their energy averaged by the cumulative distribution function and taken with the reverse sign:

$$L_{\text{ph}} = - \sum_s \langle \varepsilon_{\text{ph}} \hat{f}_s \rangle. \quad (78)$$

One can see that the Lagrangian for the phonons, definiendum (78), is invariant relative to the non-homogeneous action of the gauge group $G = SO(3) \triangleright T(3)$.

As the theory proposed by us is non-relativistic, the Lagrangian of the plasma-like medium coupling with an electromagnetic field does not contain the contribution of the latter, and its equations are considered known. Thus, the Lagrangian for a plasma-like medium with topological defects takes the form

$$\begin{aligned} L = L_C + L_e + L_{\text{ph}} = & \frac{1}{2} \rho_0 \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j + m \delta_{ij} \tilde{B}_4^j \left\langle \frac{\partial \varepsilon_e}{\partial p_i} \hat{f}_e \right\rangle + \frac{1}{2} m \langle \hat{f}_e \rangle \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j \langle \varepsilon_e \hat{f}_e \rangle \\ & - \sum_s \langle \varepsilon_{\text{ph}} \hat{f}_s \rangle - \frac{1}{2} M^{\alpha\beta\zeta\xi} E_{\alpha\beta} E_{\zeta\xi} - \frac{s_w}{2} c_{\alpha\beta} \tilde{C}_{ab}^\alpha g_w^{ac} g_w^{be} \tilde{C}_{ce}^\beta - \frac{s_Y}{2} \delta_{ij} \tilde{D}_{ab}^i g_Y^{ac} g_Y^{be} \tilde{D}_{ce}^j. \end{aligned} \quad (79)$$

3. Equations for an electromagnetic field and the conditions at the surface of moving discontinuities

As shown in [27], the most sequential way of obtaining the dynamic equations for a continuous medium coupled with an electromagnetic field is to use special relativity theory.

The phenomena considered by us are accompanied by hydrodynamic velocities which are much lower than the velocity of light in vacuum. Moreover, we consider plasma-like media which are quasi-neutral in most cases except strong discontinuities. Therefore, we assume that the equations for the electromagnetic field and the boundary conditions, including the ones at strong discontinuities, are known. (This allows us to avoid errors which can appear because of the combination of relativistic and non-relativistic terms in one Lagrangian.) Here the contribution of the electromagnetic field and other volumetric forces in the base variational equation (2) is determined by the term δW^* that takes into account the concrete requirements made on the models constructed [27].

In this section we consider the equations for the electromagnetic field, and the conditions at strong discontinuities, using exterior differential forms. Here we shall assume that the functions we consider belong to the space BV [28]. Using them, we may reduce the requirements to the smoothness of the surface of the domain occupied by the medium and of the surfaces of the internal discontinuities.

We shall define the tensors of the electromagnetic field (let us note that we consider unmagnetized and unpolarizable plasma-like media) \tilde{F}_{ab} and \tilde{F}^{ab} ($a \in I_4$) by the expressions [27] (see also [39, 45, 46])

$$\|\tilde{F}_{ab}\| = \begin{pmatrix} 0 & H^3 & -H^2 & c_0 E_1 \\ -H^3 & 0 & H^1 & c_0 E_2 \\ H^2 & -H^1 & 0 & c_0 E_3 \\ -c_0 E_1 & -c_0 E_2 & -c_0 E_3 & 0 \end{pmatrix} \quad \tilde{F}_{ab} = -\tilde{F}_{ba}$$

$$\|\tilde{F}^{ab}\| = \begin{pmatrix} 0 & H_3 & -H_2 & -E^1 c_0^{-1} \\ -H_3 & 0 & H_1 & -E^2 c_0^{-1} \\ H_2 & -H_1 & 0 & -E^3 c_0^{-1} \\ E^1 c_0^{-1} & E^2 c_0^{-1} & E^3 c_0^{-1} & 0 \end{pmatrix} \quad \tilde{F}^{ab} = -\tilde{F}^{ba}.$$

The tensor \tilde{F}_{ab} can be converted to \tilde{F}^{ab} with the help of the tensor g_f^{ab}

$$\|g_f^{ab}\| = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & c_0^{-2} \end{pmatrix}.$$

It can readily be shown [39] that

$$\tilde{F}_{ab} = \partial_a \tilde{A}_b - \partial_b \tilde{A}_a$$

where $\|\tilde{A}_a\| = \{-\mathbf{A}, c_0 \varphi\}$ is the four-dimensional vector potential of the electromagnetic field; \mathbf{A} and φ are the vector potential and the scalar potential, respectively.

Let us introduce the 2-form of the electromagnetic field by the relation

$$\tilde{F} = \frac{1}{2} \tilde{F}_{ab} dX^a \wedge dX^b. \quad (80)$$

Since there is no magnetic field source, the 2-form \tilde{F} is exact and

$$d\tilde{F} = 0. \quad (81)$$

Equation (81) contains the first pair of Maxwell equations [39]:

$$\nabla_X \cdot \mathbf{H} = 0 \quad (82)$$

$$\nabla_X \times \mathbf{E} + \frac{1}{c_0} \frac{\partial \mathbf{H}}{\partial t} = 0. \quad (83)$$

(In equations (82) and (83) ∇_X is the del in the reference configuration.)

As we deal with the class of discontinuous functions belonging to the space BV [28], the first pair of conditions at the moving surface of a discontinuity ($[\dots]$ designates the jump of an arbitrary function on the discontinuity surface; \mathbf{n} is the vector normal to the surface) follows from equations (82) and (83) [35, 36]:

$$\mathbf{n} \cdot [\mathbf{H}] = 0 \tag{84}$$

$$[\mathbf{E}_\tau] = \frac{1}{c_0} [\mathbf{H}_\tau] \times \mathbf{D}_\sigma. \tag{85}$$

In equations (84) and (85), \mathbf{D}_σ is the velocity of the discontinuity surface; τ designates the tangent direction to the discontinuity surface.

It is known [39] that the second pair of the Maxwell equations is contained in the equation

$$\partial_b \tilde{F}^{ab} = \frac{4\pi}{c_0} \tilde{J}^a \tag{86}$$

where \tilde{J}^a are the contravariant components of the four-dimensional density of the electric current. Actually, passing to a three-dimensional representation, we obtain

$$\nabla_X \cdot \mathbf{E} = 4\pi \tilde{q} \tag{87}$$

$$\nabla_X \times \mathbf{H} - \frac{1}{c_0} \frac{\partial \mathbf{E}}{\partial t} = \frac{4\pi}{c_0} \tilde{\mathbf{j}}. \tag{88}$$

In equations (87) and (88), \tilde{q} and $\tilde{\mathbf{j}}$ are the volumetric charge density and the electric current density, respectively. In a quasi-neutral plasma-like medium, the mean charge density is zero ($\tilde{q} = 0$); therefore, the displacement current (the second term on the left-hand side of equation (88)) can be neglected. This cannot be done in the consideration of the second pair of conditions at the discontinuity surface because the density of the surface charge, \tilde{q}_σ , and the density of the surface current, $\tilde{\mathbf{j}}_\sigma$, are not zero in the general case.

Let us introduce the 2-form $\tilde{\mathbf{F}}^*$ and the 3-form $\tilde{\mathbf{J}}^*$ by the relations

$$\tilde{\mathbf{F}}^* = \frac{1}{2} \tilde{F}^{ab} \mu_{ab} \quad \tilde{\mathbf{J}}^* = \tilde{J}^a \mu_a. \tag{89}$$

Then equation (86) becomes

$$d\tilde{\mathbf{F}}^* = \frac{4\pi}{c_0} \tilde{\mathbf{J}}^*. \tag{90}$$

Applying the operator of external differentiation to equation (90), we shall obtain the continuity equation for the 4-current $\tilde{\mathbf{J}}^*$, which plays the role of the integrability condition for the Maxwell equations:

$$d\tilde{\mathbf{J}}^* = 0. \tag{91}$$

The second pair of conditions at the moving surface of the discontinuity [35, 36] follows from equations (87) and (88):

$$\mathbf{n} \cdot [\mathbf{E}] = 4\pi \tilde{q}_\sigma \tag{92}$$

$$[\mathbf{H}_\tau] - \frac{1}{c_0} ([\mathbf{E}] \times \mathbf{D}_\sigma)_\tau = \frac{4\pi}{c_0} (\tilde{\mathbf{j}}_\sigma \times \mathbf{n}). \tag{93}$$

(In obtaining the last equation, we have neglected all quantities of the order of $(c_0^{-1} D_\sigma)^2$. Moreover, the surface charge and the surface current are defined in the coordinate system in which the discontinuity surface is motionless.)

The following continuity equation for the current at the moving surface of the discontinuity is equivalent to equation (91) [35]:

$$\frac{\partial \tilde{q}_\sigma}{\partial t} + \nabla_\sigma \cdot \tilde{\mathbf{j}}_\sigma + \mathbf{n} \cdot [\tilde{\mathbf{j}}] = 0. \quad (94)$$

In equation (94), $\nabla_\sigma \cdot \tilde{\mathbf{j}}_\sigma = \nabla_{\sigma,1} \tilde{j}_\sigma^1 + \nabla_{\sigma,2} \tilde{j}_\sigma^2$ is the surface divergence of the two-dimensional vector of the surface current $\tilde{\mathbf{j}}_\sigma$; $\nabla_{\sigma,\alpha}$ ($\alpha = 1, 2$) is the operator of the covariant differentiation in the plane tangent to the discontinuity surface in a given point. Equation (94) determines the flux of the charge particles ‘evaporating’ from the discontinuity surface or from the boundary of the body.

4. Obtaining the dynamic equations for a plasma-like medium with topological defects

As the starting point in obtaining the dynamic equations, we use the base variational equation (2) and Lagrangian (79).

4.1. The case of lack of strong discontinuities

4.1.1. We shall first consider the simplification of no strong discontinuity in the medium and no flux of charge particles through the boundary of the domain occupied by the medium. (The equations obtained save their form in the presence of discontinuities.) In this case the term $\delta \tilde{W}^*$ in equation (2) can be set by the expression

$$\begin{aligned} \delta \tilde{W}^* &= \int_{\Omega_4} \{(\partial_a \tilde{Q}_i^a - \tilde{Q}_j^a W_a^\alpha g_{\alpha i}^j) \delta F^i + \tilde{Q}_i^a \delta Y_a^i\} \tilde{\mu} \\ &= \int_{\Omega_4} \{(d\tilde{Q}_i + \tilde{Q}_j \wedge \tilde{G}_i^j) \delta F^i - \tilde{Q}_i \wedge \delta Y^i\} \tilde{\mu}. \end{aligned} \quad (95)$$

In equation (95) \tilde{Q}_i^a are generalized volumetric forces which contain the contribution from the fields not entering the Lagrangian (e.g., the Lorentz force); δF^i and δY_a^i are the variations of the diffeomorphism \mathbf{F} , and of the potentials of the dislocation fields, \mathbf{Y} , respectively. The second term on the variation δF^i in equation (95) points to the physical fact that the disclination fields curve the space occupied by a deformable continuum.

Let us introduce the designations as follows:

$$Z_i^a = \frac{\partial L}{\partial \tilde{B}_i^a} \quad Z_i = Z_i^a \mu_a. \quad (96)$$

Here, $Z_i^\alpha = -\sigma_i^\alpha$, σ_i^α is the Piola–Kirchhoff stress tensor [19, 34], $Z_i^4 = P_i$ is the momentum,

$$\begin{aligned} \tilde{R}_i^{ab} &= \frac{\partial L}{\partial \tilde{D}_{ab}^i} & \tilde{R}_i^{ab} &= -\tilde{R}_i^{ba} & \tilde{R}_i &= \frac{1}{2} \tilde{R}_i^{ab} \mu_{ab}. \\ \tilde{H}_\alpha^{ab} &= \frac{\partial L}{\partial \tilde{C}_{ab}^\alpha} & \tilde{H}_\alpha^{ab} &= \frac{\partial L_W}{\partial \tilde{C}_{ab}^\alpha} & \tilde{H}_\alpha^{ab} &= \tilde{H}_\alpha^{ab} + \tilde{R}_i^{ab} g_{\alpha j}^i F^j \\ \bar{H}_\alpha &= \frac{1}{2} \tilde{H}_\alpha^{ab} \mu_{ab} & \bar{H}_\alpha &= \frac{1}{2} \tilde{H}_\alpha^{ab} \mu_{ab}. \end{aligned} \quad (97)$$

In equation (98), L_W is the part of the Lagrangian (see equation (79)) containing only the disclination variables.

We shall introduce the disclination currents by the expressions [19]

$$\begin{aligned} \bar{J}_\alpha^a &= \left. \frac{\partial L}{\partial W_a^\alpha} \right|_{\tilde{C}_{ab}^\alpha} & \bar{J}^\alpha &= c^{\alpha\beta} \bar{J}_\beta & \bar{J}_\beta &= \bar{J}_\beta^a \mu_a \\ \bar{J}_\beta &\in \Lambda^3(E_4) & \bar{\mathbf{J}} &= \bar{J}^\alpha \mathbf{g}_\alpha \end{aligned} \quad (99)$$

$$\begin{aligned} \tilde{J}_\alpha^a &= \left. \frac{\partial L_W}{\partial W_a^\alpha} \right|_{\tilde{C}_{ab}^\alpha} & \tilde{J}^\alpha &= c^{\alpha\beta} \tilde{J}_\beta & \tilde{J}_\beta &= \tilde{J}_\beta^a \mu_a \\ \tilde{J}_\beta &\in \Lambda^3(E_4) & \tilde{\mathbf{J}} &= \tilde{J}^\alpha \mathbf{g}_\alpha. \end{aligned} \quad (100)$$

(In equations (98) and (99), $c^{\alpha\beta}$ are the components of the inverse Cartan–Killing metric of the subgroup $SO(3)$.)

In the component-wise form, \tilde{H}_α^{ab} , \tilde{R}_i^{ab} , \tilde{J}_α^a , and \tilde{J}_α^a look like [19]

$$\tilde{H}_\alpha^{ab} = -s_W g_W^{ca} g_W^{db} c_{\alpha\beta} \tilde{C}_{cd}^\beta \quad (101)$$

$$\tilde{R}_i^{ab} = -s_Y \delta_{ij} g_Y^{ac} g_Y^{bd} \{\partial_c Y_d^j - \partial_d Y_c^j + g_{\alpha k}^j (W_c^\alpha Y_d^k - W_d^\alpha Y_c^k) + g_{\alpha k}^j \tilde{C}_{cd}^\alpha F^k\} \quad (102)$$

$$\tilde{J}_\alpha^a = g_{\alpha j}^i (Z_i^a F^j + 2\tilde{R}_i^{ab} Y_b^j) \quad (103)$$

$$\tilde{J}_\alpha^a = 2g_{\alpha j}^i \tilde{R}_i^{ab} \tilde{B}_b^j. \quad (104)$$

Let us now obtain the dynamic equations. The base variational equation (2) in view of (95), becomes

$$\begin{aligned} \int_{\Omega_4} \delta(L\tilde{\mu})|_{F^i} + \int_{\Omega_4} \delta(L\tilde{\mu})|_{Y^i} + \int_{\Omega_4} \delta(L\tilde{\mu})|_{W^\alpha} \\ + \int_{\Omega_4} \{-(d\tilde{Q}_i + \tilde{Q}_j \wedge \tilde{G}_i^j) \delta F^i - \tilde{Q}_i \wedge \delta Y^i\} \tilde{\mu} + \delta \tilde{W} = 0. \end{aligned} \quad (105)$$

The first term in equation (105) has the form

$$\int_{\Omega_4} \delta(L\tilde{\mu})|_{F^i} = \int_{\Omega_4} \left(\frac{\partial L}{\partial F^i} - \partial_a \frac{\partial L}{\partial (\partial_a F^i)} \right) \delta F^i \tilde{\mu} + \int_{\partial\Omega_4} \frac{\partial L}{\partial (\partial_a F^i)} \delta F^i \mu_a. \quad (106)$$

Taking into account the definitions made above, we obtain

$$\begin{aligned} \frac{\partial L}{\partial (\partial_a F^i)} &= \frac{\partial L}{\partial \tilde{B}_i^a} = Z_i^a \\ \frac{\partial L}{\partial F^i} &= \frac{\partial L}{\partial \tilde{B}_i^a} \frac{\partial \tilde{B}_i^a}{\partial F^i} + \frac{\partial L}{\partial \tilde{D}_{ab}^j} \frac{\partial \tilde{D}_{ab}^j}{\partial F^i} = Z_j^a W_a^\alpha g_{\alpha i}^j + \tilde{R}_j^{ab} \tilde{C}_{ab}^\alpha g_{\alpha i}^j. \end{aligned}$$

Using these expressions and the appropriate differential forms, we can write (106) as

$$\int_{\Omega_4} \delta(L\tilde{\mu})|_{F^i} = \int_{\Omega_4} (dZ_i + Z_j \wedge \tilde{G}_i^j + 2\tilde{R}_j \wedge \tilde{T}_i^j) \delta F^i \tilde{\mu} + \int_{\partial\Omega_4} (Z_i^a \mu_a) \delta F^i. \quad (107)$$

We shall obtain the second term in (105) using arguments expressed in [19]. Let $\{\zeta^i, \quad i = 1, 2, 3\}$ be a set of arbitrary 1-forms inducing a variation of the 1-forms Y^i

$$Y^i \implies Y^i + \varepsilon \zeta^i + o(\varepsilon) \quad (108)$$

and a variation of the 2-forms \tilde{D}^i

$$\tilde{D}^i \implies \tilde{D}^i + \varepsilon (d\zeta^i + \tilde{G}_j^i \wedge \zeta^j) + o(\varepsilon). \quad (109)$$

Then an induced variation of the 4-form $L\tilde{\mu}$ takes the form

$$\begin{aligned}\delta(L\tilde{\mu})|_{Y^i} &= \left(\frac{\partial L}{\partial Y_a^i} \delta Y_a^i + \frac{\partial L}{\partial \tilde{D}_{ab}^i} \delta \tilde{D}_{ab}^i \right) \tilde{\mu} \\ &= (Z_i^a \delta Y_a^i + \tilde{R}_i^{ab} \delta \tilde{D}_{ab}^i) \tilde{\mu} = -Z_i \wedge \delta Y^i - 2\tilde{R}_i \wedge \delta \tilde{D}^i.\end{aligned}\quad (110)$$

According to equations (108) and (109), we have $\delta Y^i = \zeta^i$, $\delta \tilde{D}^i = d\zeta^i + \tilde{G}_j^i \wedge \zeta^j$. Therefore, (110) becomes

$$\begin{aligned}\delta(L\tilde{\mu})|_{Y^i} &= -Z_i \wedge \zeta^i - 2\tilde{R}_i \wedge (d\zeta^i + \tilde{G}_j^i \wedge \zeta^j) \\ &= -(Z_i + 2\tilde{R}_j \wedge \tilde{G}_i^j - 2d\tilde{R}_i) \wedge \zeta^i - 2d(\tilde{R}_i \wedge \zeta^i).\end{aligned}$$

Hence, the second term in (105) is defined by the expression

$$\int_{\Omega_4} \delta(L\tilde{\mu})|_{Y^i} = - \int_{\Omega_4} (Z_i + 2\tilde{R}_j \wedge \tilde{G}_i^j - 2d\tilde{R}_i) \wedge \delta Y^i - 2 \int_{\partial\Omega_4} \tilde{R}_i \wedge \delta Y^i.\quad (111)$$

To find the third term in (105), we may apply similar assumptions. If a set of three 1-forms $\{\eta^\alpha, \alpha = 1, 2, 3\}$ induces a variation of the potentials of the disclination fields W^α ,

$$W^\alpha \implies W^\alpha + \varepsilon \eta^\alpha + o(\varepsilon)\quad (112)$$

and a variation of the tensor of the disclination fields $\tilde{C}^\alpha = dW^\alpha + \frac{1}{2}c_{\beta\gamma}^\alpha W^\beta \wedge W^\gamma$,

$$\tilde{C}^\alpha \implies \tilde{C}^\alpha + \varepsilon(d\eta^\alpha + c_{\beta\gamma}^\alpha W^\beta \wedge \eta^\gamma) + o(\varepsilon)\quad (113)$$

then we obtain a variation of the 4-form $L\tilde{\mu}$, using equations (96)–(104) [19]:

$$\begin{aligned}\delta(L\tilde{\mu})|_{W^\alpha} &= -2\delta\tilde{C}^\alpha \wedge \tilde{H}_\alpha + \delta W^\alpha \wedge \tilde{J}_\alpha \\ &= \eta^\alpha \wedge (-2d\tilde{H}_\alpha + 2c_{\gamma\alpha}^\beta W^\gamma \wedge \tilde{H}_\beta + \tilde{J}_\alpha) - 2d(\eta^\alpha \wedge \tilde{H}_\alpha).\end{aligned}$$

This expression allows us to write the third term in equation (105) as

$$\int_{\Omega_4} \delta(L\tilde{\mu})|_{W^\alpha} = \int_{\Omega_4} \delta W^\alpha \wedge (-2d\tilde{H}_\alpha + 2c_{\gamma\alpha}^\beta W^\gamma \wedge \tilde{H}_\beta + \tilde{J}_\alpha) - 2 \int_{\partial\Omega_4} \delta W^\alpha \wedge \tilde{H}_\alpha.\quad (114)$$

Substituting equations (107), (111), and (114) into (105), we obtain

$$\begin{aligned}\int_{\Omega_4} ((d\hat{Z}_i + \hat{Z}_j \wedge \tilde{G}_i^j + 2\tilde{R}_j \wedge \tilde{T}_i^j) \delta F^i \tilde{\mu} - (\hat{Z}_i + 2\tilde{R}_j \wedge \tilde{G}_i^j - 2d\tilde{R}_i) \wedge \delta Y^i \\ + \delta W^\alpha \wedge (-2d\tilde{H}_\alpha + 2c_{\gamma\alpha}^\beta W^\gamma \wedge \tilde{H}_\beta + \tilde{J}_\alpha)) \\ - \int_{\partial\Omega_4} (-(Z_i^a \mu_a) \delta F^i + 2(\tilde{R}_i \wedge \delta Y^i) + 2(\delta W^\alpha \wedge \tilde{H}_\alpha)) + \delta \tilde{W} = 0.\end{aligned}\quad (115)$$

(In (115) we have $\hat{Z}_i = Z_i + \tilde{Q}_i$.) The requirement of stationarity of the action allows us to obtain, from (115) the desired dynamic equations and the expression for $\delta \tilde{W}$ containing the boundary conditions:

$$d\hat{Z}_i + \hat{Z}_j \wedge \tilde{G}_i^j = -2\tilde{R}_j \wedge \tilde{T}_i^j\quad (116)$$

$$d\tilde{R}_i - \tilde{R}_j \wedge \tilde{G}_i^j = \frac{1}{2}\hat{Z}_i\quad (117)$$

$$-2d\tilde{H}_\alpha + 2c_{\gamma\alpha}^\beta W^\gamma \wedge \tilde{H}_\beta + \tilde{J}_\alpha = 0\quad (118)$$

$$\delta \tilde{W} = \int_{\partial\Omega_4} (-(Z_i^a \mu_a) \delta F^i + 2(\tilde{R}_i \wedge \delta Y^i) + 2(\delta W^\alpha \wedge \tilde{H}_\alpha)).\quad (119)$$

As the surface of the domain Ω_3 occupied by the deformable continuum is not a discontinuity surface, the boundary conditions, which are similar to the relations obtained in [19], follow from equation (119):

$$(Z_i^a \delta F^i)|_{\partial\Omega_4} \mu_a|_{\partial\Omega_4} = 0 \quad (120)$$

$$\tilde{R}_i|_{\partial\Omega_4} = (\tilde{R}_i^{ab} \mu_{ab})|_{\partial\Omega_4} = 0 \quad (121)$$

$$\tilde{H}_\alpha|_{\partial\Omega_4} = (\tilde{H}_\alpha^{ab} \tilde{\mu}_\beta)|_{\partial\Omega_4} = ((\tilde{H}_\alpha^{ab} + \tilde{R}_i^{ab} g_{\alpha j}^i F^j) \tilde{\mu}_\beta)|_{\partial\Omega_4} = (\tilde{H}_\alpha^{ab} \tilde{\mu}_\beta)|_{\partial\Omega_4} = 0. \quad (122)$$

The boundary of the domain occupied by a body or a part of this boundary can be fixed. In this case the Dirichlet homogeneous boundary condition [19] holds:

$$\delta F^i|_{\partial\Omega_4} = 0. \quad (123)$$

If we have the generalized force \tilde{Q}_i on the free surface of a body (in our case it is determined by the magnetic field, which is continuous if the medium has a finite conductivity and if fluxes of charged particles from the surface are lacking), the boundary condition is equivalent to the Neumann homogeneous condition [19]

$$(Z_i^a \mu_a)|_{\partial\Omega_4} = 0. \quad (124)$$

In [19] it is shown that (118) can be rewritten relative to \tilde{H}_α :

$$d\tilde{H}_\alpha - c_{\gamma\alpha}^\beta W^\gamma \wedge \tilde{H}_\beta = \frac{1}{2} \tilde{J}_\alpha. \quad (125)$$

Using the matrices of the exterior forms, we write equations (116), (117), and (125) as

$$d\mathbf{Z} + \hat{\mathbf{Z}} \wedge \tilde{\mathbf{G}} = -2\tilde{\mathbf{R}} \wedge \tilde{\mathbf{T}} \quad (126)$$

$$d\tilde{\mathbf{R}} - \tilde{\mathbf{R}} \wedge \tilde{\mathbf{G}} = \frac{1}{2} \hat{\mathbf{Z}} \quad (127)$$

$$d\tilde{\mathbf{H}} + \tilde{\mathbf{G}} \wedge \tilde{\mathbf{H}} - \tilde{\mathbf{H}} \wedge \tilde{\mathbf{G}} = \frac{1}{2} \tilde{\mathbf{J}}. \quad (128)$$

In equation (126) we have $\hat{\mathbf{Z}} = \mathbf{Z} + \tilde{\mathbf{Q}}$. In [19] it is shown that \mathbf{Z} , $\tilde{\mathbf{R}}$, and $\tilde{\mathbf{H}}$ are transformed by the rules: ${}^*\mathbf{Z} = \mathbf{Z}\mathbf{A}^{-1}$, ${}^*\tilde{\mathbf{R}} = \tilde{\mathbf{R}}\mathbf{A}^{-1}$, ${}^*\tilde{\mathbf{H}} = \tilde{\mathbf{H}}\mathbf{A}^{-1}$. For (126) to be covariant, it is necessary to require that the matrix of generalized forces $\tilde{\mathbf{Q}}$ is transformed by the rule ${}^*\tilde{\mathbf{Q}} = \tilde{\mathbf{Q}}\mathbf{A}^{-1}$. Then (126)–(128) will acquire a convenient covariant form:

$$D\hat{\mathbf{Z}} = -2\tilde{\mathbf{R}} \wedge \tilde{\mathbf{T}} \quad (129)$$

$$D\tilde{\mathbf{R}} = \frac{1}{2} \hat{\mathbf{Z}} \quad (130)$$

$$D\tilde{\mathbf{H}} = \frac{1}{2} \tilde{\mathbf{J}}. \quad (131)$$

As for a purely dislocation material we have $\tilde{\mathbf{G}} = 0$, we obtain, instead of equations (129)–(131), the following dynamic equations:

$$d\hat{\mathbf{Z}} = 0 \quad (129')$$

$$d\tilde{\mathbf{R}} = \frac{1}{2} \hat{\mathbf{Z}}. \quad (130')$$

4.1.2. The matrices of the 3-forms $\hat{\mathbf{Z}}$, $\tilde{\mathbf{J}}$ and the matrix of the 1-forms of connectedness $\tilde{\mathbf{G}}$ entering (129)–(131) are not arbitrary. The solutions of these equations should satisfy some restrictions playing the role of integrability conditions. For (129)–(131), they are analogous to the compatibility conditions for the dynamic equations for a non-conducting isotropic continuum with topological defects, obtained in [19]:

$$DD\tilde{\mathbf{R}} = \frac{1}{2} D\hat{\mathbf{Z}} \quad DD\tilde{\mathbf{H}} = \frac{1}{2} D\tilde{\mathbf{J}}.$$

According to equations (A27) and (A28), $DD\tilde{\mathbf{R}} = \tilde{\mathbf{R}} \wedge \tilde{\mathbf{T}}$ and $DD\tilde{\mathbf{H}} = \tilde{\mathbf{T}} \wedge \tilde{\mathbf{H}} - \tilde{\mathbf{H}} \wedge \tilde{\mathbf{T}}$. Therefore, we have

$$D\hat{\mathbf{Z}} = -2\tilde{\mathbf{R}} \wedge \tilde{\mathbf{T}} \quad D\tilde{\mathbf{J}} = 2(\tilde{\mathbf{T}} \wedge \tilde{\mathbf{H}} - \tilde{\mathbf{H}} \wedge \tilde{\mathbf{T}}). \quad (132)$$

The latter of equations (132) is zero for Lagrangian (79). Thus, the integrability condition for equation (131) takes the form [19]

$$D\tilde{\mathbf{J}} = d\tilde{\mathbf{J}} + \tilde{\mathbf{G}} \wedge \tilde{\mathbf{J}} + \tilde{\mathbf{J}} \wedge \tilde{\mathbf{G}} = 0. \quad (133)$$

According to equations (100) and (104),

$$\tilde{\mathbf{J}} = -2c^{\alpha\beta} \tilde{\mathbf{R}} \mathbf{g}_\beta \wedge \tilde{\mathbf{B}} \mathbf{g}_\alpha.$$

As we have $Dc^{\alpha\beta} = 0$ and $D\mathbf{g}_\beta = 0$, equation (133) is equivalent to the relation [19]

$$D(\tilde{\mathbf{R}} \mathbf{g}_\beta \wedge \tilde{\mathbf{B}} \mathbf{g}_\alpha) = 0$$

or, in view of equations (129)–(131), to the relation

$$\hat{\mathbf{Z}} \mathbf{g}_\beta \wedge \tilde{\mathbf{B}} + 2\tilde{\mathbf{R}} \mathbf{g}_\beta \wedge D\tilde{\mathbf{B}} = \hat{\mathbf{Z}} \mathbf{g}_\beta \wedge \tilde{\mathbf{B}} + 2\tilde{\mathbf{R}} \mathbf{g}_\beta \wedge \tilde{\mathbf{D}} = 0 \quad \text{since } \tilde{\mathbf{D}} = D\tilde{\mathbf{B}}. \quad (134)$$

In [19] it is shown that $\tilde{\mathbf{R}} \mathbf{g}_\beta \wedge \tilde{\mathbf{D}} = 0$. Therefore, equation (134) takes the form

$$g_{\alpha j}^i (Z_i + \tilde{Q}_i) \wedge \tilde{B}^j = g_{\alpha j}^i (Z_i^a + \tilde{Q}_i^a) \tilde{B}_a^j = g_{\alpha j}^i ((Z_i^\beta + \tilde{Q}_i^\beta) \tilde{B}_\beta^j + (Z_i^4 + \tilde{Q}_i^4) \tilde{B}_4^j) = 0. \quad (135)$$

According to equation (96) and Lagrangian (79), the momentum and the Piola–Kirchhoff stress tensor are defined by the relations

$$\begin{aligned} Z_i^4 = P_i &= \rho_0 \delta_{ij} \tilde{B}_4^j + m \langle \hat{f}_e \rangle \delta_{ij} \tilde{B}_4^j + m \delta_{ij} \left\langle \frac{\partial \varepsilon_e}{\partial p_j} \hat{f}_e \right\rangle \\ &= \rho_0 \delta_{ij} \tilde{B}_4^j + m \langle \hat{f}_e \rangle \delta_{ij} \tilde{B}_4^j + m \langle \hat{f}_e \rangle \delta_{ij} u_C^j \end{aligned} \quad (136)$$

($u_C^j = \langle \hat{f}_e \rangle^{-1} \langle (\partial \varepsilon_e / \partial p_j) \hat{f}_e \rangle$ is the current (hydrodynamic) velocity of the conduction electrons)

$$Z_i^\alpha = -\sigma_i^\alpha = -\delta_{ij} \tilde{B}_\beta^j (\sigma_S^{\alpha\beta} + \sigma_E^{\alpha\beta} + \sigma_P^{\alpha\beta}). \quad (137)$$

In equation (137), $\sigma_S^{\alpha\beta}$ is the potential part of the stress tensor defined by the expression

$$\sigma_S^{\alpha\beta} = \frac{\partial L_S}{\partial E_{\alpha\beta}} \quad (138)$$

(L_S is the potential part of the Lagrangian). The contribution of the conduction electrons and phonons to the stress tensor is defined, respectively, as

$$\sigma_E^{\alpha\beta} = \frac{\partial \langle \varepsilon_e \hat{f}_e \rangle}{\partial E_{\alpha\beta}} \quad (139)$$

$$\sigma_{\text{Ph}}^{\alpha\beta} = \frac{\partial \Sigma_s \langle \varepsilon_{\text{ph}} \hat{f}_s \rangle}{\partial E_{\alpha\beta}}. \quad (140)$$

We shall show that $\tilde{Q}_i^\alpha = -\sigma_{M,i}^\alpha$ are the components of the stress tensor determined by the magnetic field and that $\tilde{Q}_i^4 = 0$. Then equation (135) becomes

$$g_{\alpha j}^i (-(\sigma_i^\beta + \sigma_{M,i}^\beta) \tilde{B}_\beta^j + (\rho_0 + m \langle \hat{f}_e \rangle) \delta_{ik} \tilde{B}_4^k \tilde{B}_4^j + m \langle \hat{f}_e \rangle \delta_{ik} u_C^k \tilde{B}_4^j) = 0.$$

Whence we obtain

$$g_{\alpha j}^i (\sigma_i^\beta + \sigma_{M,i}^\beta) \tilde{B}_\beta^j = 0. \quad (141)$$

For the case of no magnetic field, condition (141) coincides with an equation being equivalent to the integrability condition of the equation for the disclination fields obtained in [19] and, as shown in [19], to the balance equation for the angular momentum in a deformable non-conducting continuum with defects. Therefore, relation (141) is equivalent to the balance equation for the angular momentum in a plasma-like medium with topological defects.

Note that the integrability condition for (129) is satisfied identically, because the 5-form on the 4-space is zero [19].

4.1.3. Let us consider the energy–momentum tensor for a plasma-like medium with topological defects. This will allow us to discuss the balance equation for the energy–momentum and to find explicit expressions for the generalized forces (for the tensor \bar{Q}_i^a). According to [19], the energy–momentum tensor of a continuous medium with topological defects has the form

$$\Pi_b^a = \frac{\partial L}{\partial(\partial_a F^i)} \partial_b F^i + \frac{\partial L}{\partial(\partial_a Y_e^i)} \partial_a Y_e^i + \frac{\partial L}{\partial(\partial_a W_e^i)} \partial_a W_e^i - \delta_b^a L. \quad (142)$$

We shall present Lagrangian (79) as

$$L = L_E - s_Y \bar{L}_Y - s_W \bar{L}_W. \quad (143)$$

In equation (143), we have designated

$$\begin{aligned} L_E = & \frac{1}{2} \rho_0 \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j + m \delta_{ij} \tilde{B}_4^j \left(\frac{\partial \varepsilon_e}{\partial p_i} \hat{f}_e \right) + \frac{1}{2} m (\hat{f}_e) \tilde{B}_4^i \delta_{ij} \tilde{B}_4^j - \langle \varepsilon_e \hat{f}_e \rangle \\ & - \sum_s \langle \varepsilon_{ph} \hat{f}_s \rangle - \frac{1}{2} M^{\alpha\beta\zeta\xi} E_{\alpha\beta} E_{\zeta\xi} \end{aligned} \quad (144)$$

$$\bar{L}_Y = \frac{1}{2} \delta_{ij} \tilde{D}_{ab}^i g_Y^{ac} g_Y^{be} \tilde{D}_{ce}^j \quad (145)$$

$$\bar{L}_W = \frac{1}{2} c_{\alpha\beta} \tilde{C}_{ab}^\alpha g_W^{ac} g_W^{be} \tilde{C}_{ce}^\beta. \quad (146)$$

Expression (143) leads to the following splitting of the energy–momentum tensor:

$$\Pi_b^a = \Pi_{E,b}^a - \Pi_{Y,b}^a - \Pi_{W,b}^a. \quad (147)$$

In equation (147), designations are as follows:

$$\Pi_{E,b}^a = \frac{\partial L_E}{\partial(\partial_a F^i)} \partial_b F^i - \delta_b^a L_E = Z_i^a \partial_a F^i - \delta_b^a L_E \quad (148)$$

$$\Pi_{Y,b}^a = s_Y \left(\frac{\partial \bar{L}_Y}{\partial(\partial_a Y_e^i)} \partial_b Y_e^i - \delta_b^a \bar{L}_Y \right) = -2\tilde{R}_i^{ae} (\partial_b Y_e^i + g_{\alpha j}^i F^j \partial_b W_e^\alpha) - \delta_b^a s_Y \bar{L}_Y \quad (149)$$

$$\Pi_{W,b}^a = s_W \left(\frac{\partial \bar{L}_W}{\partial(\partial_a W_e^i)} \partial_b W_e^i - \delta_b^a \bar{L}_W \right) = -2\tilde{H}_\alpha^{ae} \partial_b W_e^i - \delta_b^a s_W \bar{L}_W. \quad (150)$$

The energy–momentum tensor, as known [19, 39], should satisfy the balance equation for the energy–momentum:

$$\partial_a \Pi_b^a = \partial_a \Pi_{E,b}^a - \partial_a \Pi_{Y,b}^a - \partial_a \Pi_{W,b}^a = 0. \quad (151)$$

We shall introduce designations $F_{E,b} = \partial_a \Pi_{E,b}^a$, $F_{Y,b} = \partial_a \Pi_{Y,b}^a$ and $F_{W,b} = \partial_a \Pi_{W,b}^a$. Then (151) becomes the balance equation for the generalized forces acting in a medium with topological defects [19]:

$$F_{E,b} = F_{Y,b} + F_{W,b}. \quad (152)$$

Let us use equations (129)–(131) for finding an explicit expression for the generalized forces. We obtain

$$F_{E,b} = \partial_a \left(\frac{\partial L_E}{\partial (\partial_a F^i)} \partial_b F^i - \delta_b^a L_E \right) = -\partial_a (\tilde{Q}_i^a \partial_a F^i) + g_{\gamma i}^j W_a^\gamma (Z_j^a + \tilde{Q}_j^a) \partial_b F^i + g_{\gamma i}^j \tilde{C}_{ac}^\gamma \tilde{R}_j^{ac} \partial_b F^i + (Z_j^a + \tilde{Q}_j^a) \partial_b (\partial_a F^i) - \partial_b L_E. \quad (153)$$

The Lagrangian L_E entering (153) depends on X^a , not only through the mapping F^i , but also through the cumulative distribution function of the quasi-particle excitations. Taking into account the kinetic equations for electrons and phonons, we obtain

$$\partial_\beta L_E = g_{\gamma i}^j W_a^\gamma Z_j^a \partial_\beta F^i \quad (154)$$

$$\partial_4 L_E = g_{\gamma i}^j W_a^\gamma Z_j^a \partial_4 F^i + \partial_\alpha (q_E^\alpha + q_P^\alpha). \quad (155)$$

In equation (155),

$$q_E^\alpha = \delta^{\alpha\beta} \partial_\beta F^i \delta_{ij} \left\langle \varepsilon_e \frac{\partial \hat{H}}{\partial p_j} \hat{f}_e \right\rangle = \delta^{\alpha\beta} \left(\left\langle \varepsilon_e \frac{\partial \varepsilon_e}{\partial p_i} \hat{f}_e \right\rangle \delta_{ij} \partial_\beta F^j + \langle \varepsilon_e \hat{f}_e \rangle \partial_4 F^i \delta_{ij} \partial_\beta F^j \right) \quad (156)$$

is the energy flux transferred by the conduction electrons,

$$q_P^\alpha = \delta^{\alpha\beta} \sum_s \left\langle \varepsilon_{\text{ph}} \frac{\partial \varepsilon_{\text{ph}}}{\partial p_i} \hat{f}_s \right\rangle \delta_{ij} \partial_\beta F^j \quad (157)$$

is the energy flux transferred by the phonons (as a rule, for a plasma-like medium (for a metal) we have $|q_P^\alpha| \ll |q_E^\alpha|$).

In deriving the dynamic equations it was mentioned that for the momentum equation be covariant, it is necessary that

$$\tilde{Q}_i^a = \frac{\partial L_M}{\partial \tilde{B}_i^a}. \quad (158)$$

In equation (158), L_M is the contribution of the magnetic field to the summarized Lagrangian of the medium and of the magnetic field L_{FM} , which we shall define below. Let us define the stress tensor of the magnetic field in the reference configuration by the relation (later it is shown that the so-defined stress tensor coincides with the deviator of the total stress tensor of the magnetic field)

$$\sigma_M^{\alpha\beta} = -\frac{\partial L_M}{\partial E_{\alpha\beta}} = \frac{1}{4\pi} H^\alpha H^\beta. \quad (159)$$

Taking into account the Maxwell equations, we obtain similarly to equations (154) and (155)

$$g_{\gamma i}^j W_a^\gamma \tilde{Q}_j^a \partial_\beta F^i = \partial_\beta L_M \quad (160)$$

$$g_{\gamma i}^j W_a^\gamma \tilde{Q}_j^a \partial_4 F^i = \partial_4 L_M - \partial_\alpha (q_M^\alpha - \tilde{Q}_i^\alpha \partial_4 F^i). \quad (161)$$

In equation (161),

$$q_M^\alpha = \frac{c_0}{4\pi} \delta^{\alpha\zeta} \partial_\zeta F^i e_{ijk} \tilde{g}_{\beta\gamma}^{jk} E^\beta H^\gamma \quad (162)$$

is the Poynting vector in the reference configuration.

Substituting equations (154), (155), (160), and (161) into (153) and taking into account that, according to [19],

$$\partial_b (\partial_a F^i) = -(\partial_b Y_a^i + g_{\gamma j}^i F^j \partial_b W_a^\gamma)$$

we obtain

$$\begin{aligned}
 \tilde{F}_{E,\beta} &= \partial_a((Z_i^a + \tilde{Q}_i^a)\partial_\beta F^i - \delta_\beta^\alpha(L_E + L_M)) \\
 &= \partial_4((P_i + \tilde{Q}_i^4)\partial_\beta F^i) - \partial_\alpha((\sigma_i^\alpha + \sigma_{M,i}^\alpha)\partial_\beta F^i + \delta_\beta^\alpha(L_E + L_M)) \\
 &= -(Z_i^a + \tilde{Q}_i^a)(\partial_b Y_a^i + g_{\gamma j}^i F^j \partial_b W_a^\gamma) + g_{\gamma i}^j \tilde{C}_{ac}^\gamma \tilde{R}_j^{ac} \partial_\beta F^i \\
 \tilde{F}_{E,4} &= \partial_a((Z_i^a + \tilde{Q}_i^a)\partial_4 F^i - \delta_4^\alpha(L_C + L_M)) + \partial_\alpha(q_E^\alpha + q_P^\alpha + q_M^\alpha - \tilde{Q}_i^\alpha \partial_4 F^i) \\
 &= \partial_4((P_i + \tilde{Q}_i^4)\partial_4 F^i - (L_C + L_M)) + \partial_\alpha(-\sigma_i^\alpha \partial_4 F^i + q_E^\alpha + q_P^\alpha + q_M^\alpha) \\
 &= -(Z_i^a + \tilde{Q}_i^a)(\partial_4 Y_a^i + g_{\gamma j}^i F^j \partial_4 W_a^\gamma) + g_{\gamma i}^j \tilde{C}_{ac}^\gamma \tilde{R}_j^{ac} \partial_4 F^i.
 \end{aligned}$$

As the theory developed by us is non-relativistic, we have $\tilde{Q}_i^4 \equiv 0$ and therefore

$$L_M = -\frac{H^2}{8\pi} = -\frac{1}{8\pi} \delta_{ij} H^i H^j = -\frac{1}{8\pi} \tilde{g}_{\alpha\beta} H^\alpha H^\beta. \quad (163)$$

From expression (159) it follows that

$$\sigma_M^{\alpha\beta} = -\frac{\partial L_M}{\partial E_{\alpha\beta}} = -2\frac{\partial L_M}{\partial \tilde{g}_{\alpha\beta}} = \frac{1}{4\pi} H^\alpha H^\beta.$$

Finally, we obtain

$$\begin{aligned}
 \tilde{F}_{E,\beta} &= \partial_4(P_i \partial_\beta F^i) - \partial_\alpha((\sigma_i^\alpha + \sigma_{M,i}^\alpha)\partial_\beta F^i + \delta_\beta^\alpha(L_E + L_M)) \\
 &= -(Z_i^a + \tilde{Q}_i^a)(\partial_b Y_a^i + g_{\gamma j}^i F^j \partial_b W_a^\gamma) + g_{\gamma i}^j \tilde{C}_{ac}^\gamma \tilde{R}_j^{ac} \partial_\beta F^i \quad (164)
 \end{aligned}$$

$$\begin{aligned}
 \tilde{F}_{E,4} &= \partial_4(P_i \partial_4 F^i - (L_E + L_M)) + \partial_\alpha(-\sigma_i^\alpha \partial_4 F^i + q_E^\alpha + q_P^\alpha + q_M^\alpha) \\
 &= -(Z_i^a + \tilde{Q}_i^a)(\partial_4 Y_a^i + g_{\gamma j}^i F^j \partial_4 W_a^\gamma) + g_{\gamma i}^j \tilde{C}_{ac}^\gamma \tilde{R}_j^{ac} \partial_4 F^i. \quad (165)
 \end{aligned}$$

Acting similarly, we shall obtain expressions for $F_{Y,b}$ and $F_{W,b}$ as

$$\begin{aligned}
 F_{Y,b} &= \partial_a \Pi_{Y,b}^a = -(Z_i^a + \tilde{Q}_i^a)(\partial_b Y_a^i + g_{\gamma j}^i F^j \partial_b W_a^\gamma) - g_{\alpha i}^k \tilde{R}_k^{ac} \{2W_e^\alpha (\partial_b Y_c^i + g_{\beta j}^i F^j \partial_b W_c^\beta) \\
 &\quad - \partial_b (W_e^\alpha Y_c^i - W_c^\alpha Y_e^i) + \partial_e (2F^i \partial_b W_c^\alpha) - \partial_b (F^i \tilde{C}_{ec}^\alpha)\} \quad (166)
 \end{aligned}$$

$$F_{W,b} = \partial_a \Pi_{W,b}^a = -\tilde{J}_\alpha^a \partial_b W_a^\alpha + c_{\beta\gamma}^\eta \{2c_{\alpha\eta} c^{\gamma\delta} \tilde{H}_\delta^{ac} W_a^\beta \partial_b W_c^\alpha + \tilde{H}_\eta^{ac} \partial_b (W_a^\beta W_c^\gamma)\}. \quad (167)$$

We shall note that equation (167) is equivalent to an analogous relation obtained in [19], since the appropriate dynamic equations for disclination fields are identical. Comparison of equations (164) and (165) with (166) and (167) shows that the action of the elastic forces and of the magnetic field on the topological defects is completely compensated. Therefore, only interactions between the topological defects contribute to the balance equation for the forces. (The fact that the balance equation for the forces involves no force of interaction with the magnetic field is completely coordinated with the fact that the media considered by us are unmagnetized and unpolarizable.) Hence, the final form for the balance equation of forces acting in a medium with topological defects and a magnetic field, the total energy-momentum tensor of which looks like

$$\begin{aligned}
 \tilde{\Pi}_\beta^\alpha &= \Pi_\beta^\alpha + \tilde{Q}_i^\alpha \partial_\beta F^i - \delta_\beta^\alpha L_M = (Z_i^\alpha + \tilde{Q}_i^\alpha) \partial_\beta F^i + 2\tilde{R}_i^{\alpha e} (\partial_\beta Y_e^i + g_{\gamma j}^i F^j \partial_\beta W_e^\gamma) \\
 &\quad + 2\tilde{H}_\gamma^{\alpha e} \partial_\beta W_e^\gamma - \delta_\beta^\alpha (L_E + L_M - s_Y \bar{L}_Y - s_W \bar{L}_W) \quad (168)
 \end{aligned}$$

$$\tilde{\Pi}_\beta^4 = \Pi_\beta^4 + \tilde{Q}_i^4 \partial_\beta F^i = P_i \partial_\beta F^i + 2\tilde{R}_i^{4e} (\partial_\beta Y_e^i + g_{\gamma j}^i F^j \partial_\beta W_e^\gamma) + 2\tilde{H}_\gamma^{4e} \partial_\beta W_e^\gamma \quad (169)$$

$$\tilde{\Pi}_4^\alpha = Z_i^\alpha \partial_4 F^i + q_E^\alpha + q_P^\alpha + q_M^\alpha + 2\tilde{R}_i^{\alpha e} (\partial_4 Y_e^i + g_{\gamma j}^i F^j \partial_4 W_e^\gamma) + 2\tilde{H}_\gamma^{\alpha e} \partial_4 W_e^\gamma \quad (170)$$

$$\tilde{\Pi}_4^4 = P_i \partial_4 F^i + 2\tilde{R}_i^{4e} (\partial_4 Y_e^i + g_{\gamma j}^i F^j \partial_4 W_e^\gamma) + 2\tilde{H}_\gamma^{4e} \partial_4 W_e^\gamma - (L_E + L_M - s_Y \bar{L}_Y - s_W \bar{L}_W) \quad (171)$$

coincides with the balance equation for the forces acting in a deformable isotropic continuum with defects, non-interacting with the magnetic field and the quasi-particle excitations [19]

$$\begin{aligned} \tilde{F}_{E,b} - F_{Y,b} - F_{W,b} &= g_{\gamma i}^j \tilde{C}_{ac}^\gamma \tilde{R}_j^{ac} \partial_b F^i + g_{\alpha i}^k \tilde{R}_k^{ec} [2W_e^\alpha (\partial_b Y_c^i + g_{\beta j}^i F^j \partial_b W_c^\beta) \\ &\quad - \partial_b (W_e^\alpha Y_c^i - W_c^\alpha Y_e^i) + \partial_e (2F^i \partial_b W_c^\alpha) - \partial_b (F^i \tilde{C}_{ec}^\alpha)] + \tilde{J}_\alpha^a \partial_b W_a^\alpha \\ &\quad + c_{\beta\gamma}^\eta [2c_{\alpha\eta} c^{\gamma\delta} \tilde{H}_\delta^{ac} W_a^\beta \partial_b W_c^\alpha + \tilde{H}_\eta^{ac} \partial_b (W_a^\beta W_c^\gamma)] = 0. \end{aligned} \quad (172)$$

Equation (172) shows that for purely dislocation materials the balance equation for the energy–momentum is satisfied identically.

4.2. Conditions at strong discontinuities

Let us obtain the conditions at strong hydrodynamic discontinuities (conditions at strong electromagnetic discontinuities are given in section 3 (equations (84), (85), (92) and (93) together with the continuity equation for the electric current (94)). These conditions are obtained either with the help of passages to the limit in the balance equations for masses, momenta, and energy in the integrated form [33], or with the help of the respective balance equations in the differential form written in the class of discontinuous functions belonging to the space BV [28]. In [26, 27] it is suggested that the base variational equation (2) is used not only for obtaining the dynamic equations and the boundary conditions, but also for obtaining the conditions at the surfaces of strong discontinuities. (Later we shall use both methods, so thus our aim is to obtain both the conditions at the discontinuity for the forms Z_i , \tilde{R}_i , and \tilde{H}_α (\tilde{H}_α) and an equivalent for the integrability condition (141).)

We write the balance equation for the energy–momentum tensor, taking into account equations (166)–(169), as two equations:

$$\partial_4 \tilde{\Pi}_\beta^4 + \partial_\alpha \tilde{\Pi}_\beta^\alpha = 0 \quad (173)$$

$$\partial_4 \tilde{\Pi}_4^4 + \partial_\alpha \tilde{\Pi}_4^\alpha = 0. \quad (174)$$

The last expression is the energy balance equation for a current-carrying continuous medium with topological defects. We shall note that in the case of the presence of quasi-particle excitations (conduction electrons, phonons, etc) (174) should be considered simultaneously with (129)–(131).

Let the functions considered in the present subsection belong to the space BV [28]. Then (173) and (174) are equivalent to the following conditions at the surface of a strong discontinuity Σ :

$$[\tilde{\Pi}_\beta^\alpha n_\alpha] - [\tilde{\Pi}_\beta^4] D_N = 0 \quad (175)$$

$$[\tilde{\Pi}_4^\alpha n_\alpha] - [\tilde{\Pi}_4^4] D_N = 0. \quad (176)$$

In equations (175) and (176), $D_N = \mathbf{D}_\sigma \cdot \mathbf{n}$ is the normal velocity component at the discontinuity surface. If the equation of the surface Σ has the form $\mathfrak{S}(X^\alpha, t) = \mathfrak{S}(X^\alpha, X^4) = 0$, then \mathbf{D}_σ is defined by the relation [35]

$$\mathbf{D}_\sigma = -\frac{\partial_4 \mathfrak{S}}{|\nabla_X \mathfrak{S}|} \mathbf{n}. \quad (177)$$

Preparatory to obtaining conditions at the strong discontinuity with the help of the base variational equation (2), we shall make a few remarks on the balance of the angular momentum at the discontinuity surface Σ . Equation (141), being the integrability condition for (131) and, simultaneously, the balance equation for the angular momentum in the

medium with topological defects, is correct on both sides of the discontinuity. Therefore, we have at the discontinuity

$$(g_{\alpha j}^i(\sigma_i^\beta + \sigma_{M,i}^\beta)\tilde{B}_\beta^j)_- = (g_{\alpha j}^i(\sigma_i^\beta + \sigma_{M,i}^\beta)\tilde{B}_\beta^j)_+ = 0.$$

Hence, there is no need to consider the balance equation for the angular momentum at the strong discontinuity.

Besides the balance equation of the energy–momentum at the discontinuity surface, it is necessary to obtain conditions for the dislocation and the disclination fields. This can be done only with the help of the base variational equation (2). It is obvious that the relation for the momentum balance at a strong discontinuity, which can be obtained with the help of (2), should coincide with (175). For this purpose it is necessary to introduce respective alterations in expression (113) for δW^* . (We have the right to do this because the given term in the base variational equation refers to the assigned terms in constructing models of continuous media [25–27].)

Let us define δW^* by the relation

$$\begin{aligned} \delta \tilde{W}^* = & \int_{\Omega_4} \{ (d\tilde{Q}_i + \tilde{Q}_j \wedge \tilde{G}_i^j) \delta F^i - \tilde{Q}_i \wedge \delta Y^i \} \tilde{\mu} \\ & - \int_{\partial\Omega_4 + \Sigma^\pm} \{ \tilde{Q}_i^a + (2\tilde{R}_j^{ab} \partial_\beta Y_a^j + 2\tilde{H}_\gamma^{ab} \partial_\beta W_a^\gamma - \delta_\beta^a L) (\partial_\beta F^i)^{-1} \} \mu_a \delta F^i. \end{aligned}$$

Then expression (119) for δW becomes

$$\begin{aligned} \delta \tilde{W} = & \int_{\partial\Omega_4 + \Sigma^\pm} (- (Z_i^a + \tilde{Q}_i^a + (2\tilde{R}_j^{ab} \partial_\beta Y_a^j + 2\tilde{H}_\gamma^{ab} \partial_\beta W_a^\gamma - \delta_\beta^a L) (\partial_\beta F^i)^{-1} L) \mu_a \delta F^i \\ & + 2\tilde{R}_i^{ab} \mu_b \delta Y_a^i + 2\tilde{H}_\alpha^{ab} \mu_b \delta W_a^\alpha) \\ = & \int_{\partial\Omega_4 + \Sigma^\pm} (- \tilde{\Pi}_\beta^a (\partial_\beta F^i)^{-1} \mu_a \delta F^i + 2\tilde{R}_i^{ab} \mu_b \delta Y_a^i + 2\tilde{H}_\alpha^{ab} \mu_b \delta W_a^\alpha). \end{aligned}$$

Hence we obtain the conditions at the strong discontinuity surface, taking into account the fact that the variations δF^i , δY_a^i , and δW_a^α are independent and continuous (approximate continuous [28]):

$$[\tilde{\Pi}_\beta^a (\partial_\beta F^i)^{-1} \mu_a] = 0 \tag{178}$$

$$[\tilde{R}_i^{ab} \mu_b] = -[\tilde{R}_i^{a4}] D_N + [\tilde{R}_i^{a\beta} n_\beta] = 0 \tag{179}$$

$$[\tilde{H}_\alpha^{ab} \mu_b] = -[\tilde{H}_\alpha^{a4}] D_N + [\tilde{H}_\alpha^{a\beta} n_\beta] = 0. \tag{180}$$

As the mapping $F^i(X^a)$ is the integrated response of the system to a perturbation, it should be expected that it is approximately continuous at the discontinuity surface. In this case the desired condition for the momentum balance at a strong discontinuity follows from (178):

$$[\tilde{\Pi}_\beta^a \mu_a] = [\tilde{\Pi}_\beta^\alpha n_\alpha] - [\tilde{\Pi}_\beta^4] D_N = 0. \tag{181}$$

Besides the conditions at a strong discontinuity obtained above, the mass balance should hold. In our case, the evolution of the mass density $\hat{\rho} = \rho_0 + m \langle \hat{f}_e \rangle$ in the reference configuration is determined by the flux $\tilde{J}_M^\alpha = \delta^{\alpha\beta} \tilde{\Pi}_\beta^4$. Introducing the mass 4-current $\|\tilde{J}_M^a\| = [\tilde{J}_M^1, \tilde{J}_M^2, \tilde{J}_M^3, \hat{\rho}]^T$, we can write the continuity equation for this current, expressing the invariance of the total mass of the deformable continuum:

$$d\tilde{J}_M = 0 \quad \tilde{J}_M = \tilde{J}_M^a \mu_a \tag{182}$$

or

$$\partial_4 \hat{\rho} + \partial_\alpha \tilde{J}_M^\alpha = 0. \tag{183}$$

Assuming, as earlier, that we deal with discontinuous functions belonging to the space BV , we obtain from (183) the mass balance condition at the strong discontinuity:

$$-[\hat{\rho}]D_N + [\tilde{J}_M^\alpha n_\alpha] = 0. \quad (184)$$

5. Discussion

The phenomenological characteristics of the topological defects (distortion \tilde{b}_α^i , velocity \tilde{V}^i , dislocation density $n_\varphi^{\alpha i}$, dislocation flux $J_{\varphi,\alpha}^i$, spin \tilde{w}_α^i , bend torsion $\tilde{k}^{\alpha i}$, disclination density n_w^i , and disclination flux J_w^i) can be expressed through the fields F^i , W_α^α , and Y_a^i with the help of equations (23), (24), (31), (32), and (44)–(47):

$$\tilde{b}_\alpha^i = \partial_\alpha F^i + g_{\beta j}^i F^j W_\alpha^\beta + Y_\alpha^i \quad (185)$$

$$\tilde{V}^i = \partial_4 F^i + g_{\beta j}^i F^j W_4^\beta + Y_4^i \quad (186)$$

$$J_{\varphi,\alpha}^i = g_{\beta j}^i (W_\alpha^\beta Y_4^j - W_4^\beta Y_\alpha^j + F^j (\partial_\alpha W_4^\beta - \partial_4 W_\alpha^\beta + c_{\gamma\kappa}^\beta W_\alpha^\gamma W_4^\kappa)) + \partial_\alpha Y_4^i - \partial_4 Y_\alpha^i \quad (187)$$

$$n_\varphi^{\alpha i} = e^{\alpha\zeta\xi} (\partial_\zeta Y_\xi^i - \partial_\xi Y_\zeta^i + g_{\beta j}^i (W_\zeta^\beta Y_\xi^j - W_\xi^\beta Y_\zeta^j + F^j (\partial_\zeta W_\xi^\beta - \partial_\xi W_\zeta^\beta + c_{\gamma\kappa}^\beta W_\zeta^\gamma W_\xi^\kappa))) \quad (188)$$

$$\tilde{w}_\alpha^i = W_\alpha^\zeta g_{\zeta j}^i (\partial_4 F^j + W_4^\kappa g_{\kappa l}^j F^l + Y_4^j) \quad (189)$$

$$\tilde{k}^{\alpha i} = e^{\alpha\beta\gamma} W_\beta^\zeta g_{\zeta j}^i (\partial_\gamma F^j + W_\gamma^\kappa g_{\kappa l}^j F^l + Y_\gamma^j) \quad (190)$$

$$J_w^i = e^{\alpha\zeta\xi} (\partial_\zeta (W_\xi^\beta Y_4^j - W_4^\beta Y_\xi^j + \tilde{C}_{\xi 4}^\beta F^j) + \partial_4 (W_\zeta^\beta Y_\xi^j - W_\xi^\beta Y_\zeta^j + \tilde{C}_{\zeta\xi}^\beta F^j)) g_{\beta j}^i \quad (191)$$

$$n_w^i = e^{\alpha\zeta\xi} \partial_\alpha (W_\zeta^\beta Y_\xi^j - W_\xi^\beta Y_\zeta^j + \tilde{C}_{\zeta\xi}^\beta F^j) g_{\beta j}^i. \quad (192)$$

Equations (185)–(192) are substantially simplified for a purely dislocation material

$$\tilde{b}_\alpha^i = \partial_\alpha F^i + Y_\alpha^i \quad (193)$$

$$\tilde{V}^i = \partial_4 F^i + Y_4^i \quad (194)$$

$$J_{\varphi,\alpha}^i = \partial_\alpha Y_4^i - \partial_4 Y_\alpha^i \quad (195)$$

$$n_\varphi^{\alpha i} = e^{\alpha\zeta\xi} (\partial_\zeta Y_\xi^i - \partial_\xi Y_\zeta^i). \quad (196)$$

It is known [19, 21] that the Burgers vector (\mathbf{a}_B) for closed curves and the Frank vector (\mathbf{a}_F) for closed two-dimensional surfaces are observable variables in the theory of topological defects. According to [19], we have

$$a_F^i(\partial\Omega_3) = \int_{\partial\Omega_3} (\tilde{T}_j^i F^j)|_{t=\text{const}} + \int_{\partial\Omega_3} (\tilde{G}_j^i \wedge Y^j)|_{t=\text{const}} \quad (197)$$

$$a_B^i(\partial\Sigma_2) = \int_{\partial\Sigma_2} Y^i|_{t=\text{const}} + \int_{\partial\Sigma_2} \hat{H}_3(\tilde{G}_j^i F^j)|_{t=\text{const}}. \quad (198)$$

In equations (197) and (198), Ω_3 is a closed domain of the three-dimensional space occupied by a continuous medium with boundary $\partial\Omega_3$ which is a closed two-dimensional surface in E_3 ; Σ_2 is a closed two-dimensional surface with boundary $\partial\Sigma_2$ which is a closed curve.

For a purely dislocation material ($W^\alpha \equiv 0$), the following equalities hold [19]:

$$\begin{aligned} a_F^i(\partial\Omega_3) &= 0 & \text{for } \forall \partial\Omega_3 \\ a_B^i(\partial\Sigma_2) &= \int_{\partial\Sigma_2} Y^i|_{t=\text{const}} = \int_{\Sigma_2} dY^i|_{t=\text{const}} & \text{for } \forall \partial\Sigma_2. \end{aligned} \quad (199)$$

From equations (197) and (198) it follows that the exact part of the 1-forms of the velocity distortion do not contribute to the Burgers vector $a_B^i(\partial\Sigma_2)$ for the 1-curves and to the Frank vector $a_F^i(\partial\Omega_3)$ for the 2-curves. However, the functions $F^i(X^\zeta, t)$, as shown in

[19], determine identically a current configuration of the body through the coordinates $\{X^\zeta\}$ of the reference configuration. Let [19]

$$\eta : [0, 1] \implies E_4 | \bar{X}^\alpha = \lambda X^\alpha \quad \bar{t} = \lambda t \quad \text{for } \forall \lambda \in [0, 1] \quad (200)$$

be the line (mapping) connecting the point $P_0(0, 0, 0, 0)$ with the point $P_1(X^1, X^2, X^3, t)$ in E_4 . This mapping induces the reversible relations [19]

$$\eta^* d\bar{X}^\alpha = X^\alpha d\lambda \quad \eta^* d\bar{t} = t d\lambda$$

for the 1-forms on the interval $[0, 1]$ for each point $P^1(X^1, X^2, X^3, t)$ in E_4 . Defining $x^i(X^a)$ as integrals over paths (200) [19],

$$x^i(X^a) = \int_{[0,1]} \eta^* \tilde{B}^i$$

we shall obtain

$$\begin{aligned} x^i(X^\zeta, t) &= \int_0^1 \{\partial_a F^i + W_a^\alpha g_{\alpha j}^i F^j + Y_a^i\}(\lambda X^\zeta, \lambda t) X^a d\lambda \\ &= F^i(X^\zeta, t) - F^i(0^\zeta, 0) + \xi^i \end{aligned}$$

where

$$\xi = \int_0^1 X^a \{W_a^\alpha g_{\alpha j}^i F^j + Y_a^i\}(\lambda X^\zeta, \lambda t) d\lambda. \quad (201)$$

As we have $X^a = \lambda X^a(1/\lambda)$, equation (202) takes the form

$$\xi = \int_0^1 \{X^a W_a^\alpha g_{\alpha j}^i F^j + X^a Y_a^i\}(\lambda X^\zeta, \lambda t) \frac{d\lambda}{\lambda}.$$

Because the 1-forms Y^i and W_a^α are inexact forms (they satisfy the inexact gauge conditions (12) in any point $X^a \in E_4$, including the point $\{\lambda X^\zeta, \lambda t\}$ for $\forall \lambda \in [0, 1]$), $\xi^i = 0$ and

$$x^i(X^\zeta, t) = F^i(X^\zeta, t) - F^i(0^\zeta, 0). \quad (202)$$

Thus the mapping of the reference configuration of a body with defects is realized in the current configuration as an integration of the 1-forms of the distortion along the lines (200), whereas in the case of defectless materials, the path of integration from the point P_0 to the point P_1 is arbitrary:

$$x^i(X^a) = F^i(X^a) - F^i(0^a) = \int_{P_0}^{P_1} \tilde{B}^i = \int_{P_0}^{P_1} dF^i.$$

(In [19] it is noted that the lines (200) are a generalization of the virtual quasi-static processes considered in thermodynamics.)

The homotopy operator (A9) introduces, in the calculation of the field characteristics of topological defects (see (185)–(192)), integrated operators of the type [19]

$$I_k \langle \varphi \rangle (X^\alpha, t) = \int_0^1 \lambda^k \varphi(\lambda X^\alpha, \lambda t) d\lambda \quad (203)$$

($\varphi(X^\alpha, t)$ is an arbitrary field variable). It is obvious that the operators (203) make the theory of topological defects interacting with quasi-particle excitations and with the magnetic field non-local in time and space. Thus, from equation (125) it follows that a direct coupling between the disclination fields and the magnetic field is formally lacking. This coupling is indirect as the tensor \tilde{H}_α^{ab} is determined by the tensor \tilde{R}_i^{ab} and by the 1-forms of the distortion velocity \tilde{B}^i (see (110)), the spacetime evolution of which is described by (123) and

(124). (According to these equations, the momentum and summarized stress tensor of the medium and of the magnetic field, containing the contribution of quasi-particle excitations, are the source of the dislocation fields.)

The way of introducing quasi-particle excitations (phonons and conduction electrons) present in a medium with topological defects through the construction of the minimal substitution in the form of equations (67)–(69) for electrons and (76) and (77) for phonons which we have offered is practically equivalent to the well known approximation of the mean field. Deviations (fluctuations) from the mean field can be taken into account with the help of the operators of electron–phonon and phonon–electron collisions. (To construct their models a so-called diffraction model of metal [47], which is applicable for metals with defects, and the concepts expressed by us in [48], in obtaining the transport coefficients for defectless plasma-like media can be used. This problem will be considered in other work.)

Note that the ground state (or the vacuum state relative to the quasi-particle excitations of a the medium with defects) depends on the mean energy of excitations similar to how in quantum field theory it is determined at finite temperatures by the temperature [49]. In our case, this is equivalent to the appearance of a dependence of the mass density ρ_0 on the mean energy of quasi-particle excitations even in a defectless material. In the presence of topological defects, the ground state depends not only on the mean energy of quasi-particle excitations, but also on the field characteristics of the defects. Moreover, the present theory of topological defects interacting with quasi-particle excitations and with a magnetic field is nonlinear and non-local. Therefore, it is possible to induce instabilities leading to a break of the unique connection, in the sense of (202), between the reference and the actual configuration. In this case, the current state of a deformable continuum with topological defects will be determined by more than one reference configuration with incommensurable symmetries. It is obvious that a more common description of the dynamics of a deformable medium with topological defects interacting with quasi-particle excitations should take into account the mentioned situation from the very beginning. (It is planned to consider one possible way of solving this problem in another work.)

We have used a method of obtaining the dynamic equations which, in view of the non-relativistic character of motions of a continuous medium and of the relativistic Maxwell equations, represents a combination of the variational principle in the form offered in [26, 27] and of the requirement according to which the balance equation for the total energy–momentum tensor of a plasma-like medium and of the magnetic field should be satisfied. (Note that the requirement of satisfaction of the conservation laws, in particular, the energy balance equation, since the Landau pioneering work on the theory of superfluidity of helium II [50, 51], is widely used to construct continuous models of quantum [37, 52–54] and classical [55–57] multivelocitity continua.) The use of the base variational equation (2) has allowed us to obtain a gauge-covariant system of the dynamic equations (129)–(131) and conditions at strong discontinuities (179)–(182). The requirement of satisfaction of the balance equation for the total energy–momentum tensor of the plasma-like medium with defects and of the magnetic field has offered us a non-contradictory way to include the coupling between the magnetic field and defects in the present theory. An investigation of the integrability condition of the equation for disclination fields has shown that it is equivalent to the balance equation for the angular momentum in a plasma-like medium with topological defects. In the case of lack of a magnetic field and defects, this condition is degenerated into the requirement of symmetry of the Cauchy stress tensor. An analysis of the force balance in the present theory has shown that the magnetic field does not make a direct contribution to the interactions between topological defects which correlates with the fact that the plasma-like media considered are unmagnetized and unpolarizable.

Thus, the system of dynamic equations (129)–(131) obtained, together with the condition of integrability (161), the energy balance equation (194), the kinetic equations for electrons and phonons, and the Maxwell equations, with the appropriate conditions at the boundary of the domain occupied by the medium and at the strong discontinuity surfaces, is self-consistent and closed and can be used to solve the physical problems formulated in the introduction.

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Appendix. The exact and inexact differential forms

Following [19] (also see [31, 32, 40, 41, 58, 59]), we shall consider the minimum information about the exterior differential forms that is necessary for our aims. In addition, we assume that the considered functions belong to the space BV containing discontinuous functions, the first generalized derivatives of which are measures [28].

The ‘physical’ space in which an evolution of a continuous medium proceeds is an Euclidean space (Euclidean manifold) E_4 with the Cartesian coordinate covering $\{X^a, a \in I_4 = \{1, 2, 3, 4\}\}$ (coordinate X^4 corresponds to time t). Let us agree to use the lower-case greek $\alpha, \beta, \gamma, \dots$ and latin characters, beginning from i , for the designation of indices from the set I_3 . The lower-case latin characters from a to h , similarly to [19], we use for the designation of indices from the set I_4 . Thus, as usually, the presence in the formulae of repeating or umbral indices points to the fact that a summation is performed over these indices.

We shall designate the set of all functions $\varphi(X^a) \in BV(E_4)$ as Λ^0 . These functions are scalars or 0-forms. (Below we shall assume that $BV(E_4) = E_4 = BV$.) The exterior differential forms of degree k (k -forms) defined on $BV(E_4)$ will be designated as $\Lambda^k(E_4)$. The set $\Lambda^4(E_4)$, defined on BV represents the one-dimensional vector space with a natural base $\{\tilde{\mu}\}$. We shall designate the element of volume in E_4 as

$$\tilde{\mu} = dX^1 \wedge dX^2 \wedge dX^3 \wedge dX^4 = \frac{1}{4!} e_{abcf} dX^a \wedge dX^b \wedge dX^c \wedge dX^f. \quad (A1)$$

In equation (A1), e_{abcf} is a component of the Levi-Civita tensor, and the external product is designated by \wedge . Let us designate the three-dimensional element of volume in space E_3 as $\mu = dX^1 \wedge dX^2 \wedge dX^3$. Therefore, we have $\tilde{\mu} = \mu \wedge dt$.

In differential geometry, a $T(E_4)$ space tangential to E_4 is introduced for which the derivatives $\{\partial_a = \partial/\partial X^a\}$, understood as measures, can be used as a natural base. The element $v \in T(E_4)$ is determined by the expression $v = v^a(X^b)\partial_a$, where X^b is a contact point of space $T(E_4)$ to E_4 . (As an example of a space tangential to E_4 it is possible to cite the vector field of the velocities of the points of a deformable continuous medium.) The dual base of the space $T(E_4)$ for the base $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ is the natural base of the four-dimensional space $\Lambda^1(E_4)$ of all 1-forms defined on $BV\{dX^1, dX^2, dX^3, dX^4\}$.

The inverse (bijective) generating base of the vector space of the 3-forms $\Lambda^3(E_4)$ of dimension $\binom{4}{3}$ has the form [19]

$$\mu_a = \partial_a \cdot \tilde{\mu} = \frac{1}{3!} e_{abcf} dX^b \wedge dX^c \wedge dX^f. \tag{A2}$$

In equation (A2) and below, the symbol ‘ \cdot ’ designates an internal (scalar) product. Introducing the operator of external differentiation $d = dX^a \wedge \partial_a$, it can be shown [19] that the base μ_a satisfies the equalities

$$d\mu_a = 0 \quad dX^a \wedge \mu_b = \delta_b^a \tilde{\mu}. \tag{A3}$$

($\delta_b^a = 1$ at $a = b$; $\delta_b^a = 0$ at $a \neq b$.)

According to [19], the bijective base of the $\binom{4}{2}$ -dimensional space of all 2-forms $\Lambda^2(E_4)$ is given by

$$\mu_{ab} = \partial_a \cdot \mu_b = \partial_a \cdot (\partial_b \cdot \tilde{\mu}) \quad \text{for } a < b. \tag{A4}$$

This base has the following properties [19]:

$$\mu_{ab} = -\mu_{ba} \quad d\mu_{ab} = 0 \quad dX^c \wedge \mu_{ab} = \delta_a^c \mu_b - \delta_b^c \mu_a. \tag{A5}$$

In equations (A3)–(A5), d is the four-dimensional operator of external differentiation:

$$d = dX^a \wedge \partial_a = dX^\alpha \wedge \partial_\alpha + dt \wedge \partial_4 = \bar{d} + dt \wedge \partial_4. \tag{A6}$$

$\bar{d} = dX^\alpha \wedge \partial_\alpha$ is the three-dimensional operator of external differentiation.

From equation (A6) we have the following equalities [19]:

$$\partial_4 \cdot \tilde{\mu} = 0 \quad \partial_4 \cdot dt = 1 \quad \partial_\alpha \cdot dt = 0 \quad \partial_\alpha \cdot \tilde{\mu} = \tilde{\mu}_\alpha. \tag{A7}$$

In equation (A7), $\{\tilde{\mu}_\alpha\}$ is the bijective generating base of the $\binom{3}{2}$ -dimensional space $\Lambda^2(E_3)$ of all 2-forms defined on $BV(E_3)$. Then we have

$$\mu_a = (\partial_a \cdot \mu) \wedge dt - \mu \wedge (\partial_a \wedge dt) = \delta_a^\alpha \tilde{\mu}_\alpha \wedge dt - \delta_a^4 \mu. \tag{A8}$$

The bases introduced by equations (A2)–(A8) allow us to define arbitrary k -forms. For example, the arbitrary 3-form $K \in BV(BV \subseteq E_4)$ is uniquely defined in a base of the $\binom{4}{3}$ -dimensional space $\Lambda^3(E_4)$ by the expression

$$K = K^a \mu_a = K^\alpha \mu_\alpha + K^4 \mu_4 = K^\alpha \tilde{\mu}_\alpha \wedge dt - K^4 \mu$$

with its differential given by the relation

$$dK = (\bar{d}(K^\alpha \tilde{\mu}_\alpha) + \partial_4 K^4) \wedge dt = (\partial_\alpha K^\alpha + \partial_4 K^4) \mu \wedge dt.$$

Accordingly, the arbitrary 2-form o defined on $BV(E_4)$ can uniquely be expressed through the 1-form $r = r_\alpha dX^\alpha$ and the 2-form $h = h^\alpha \tilde{\mu}_\alpha$ [19]:

$$o = r \wedge dt + h = r_\alpha dX^\alpha \wedge dt + h^\alpha \tilde{\mu}_\alpha.$$

Its external differential has the form

$$do = (\bar{d}r + \partial_4 h) \wedge dt + \bar{d}h = (\bar{d}(r_\alpha dX^\alpha) + (\partial_4 h^\alpha) \tilde{\mu}_\alpha) \wedge dt + \bar{d}(h^\alpha \tilde{\mu}_\alpha).$$

For the construction of models of continuous media with topological defects it is essential that the operator of an external differentiation leads to a splitting of the space $\Lambda(E_4)$ into two subspaces, one of which contains all exact (closed) forms. (The element

$o \in \Lambda(E_4)$ is closed if only if $do = 0$. The element $o \in \Lambda(E_4)$ is named exact if there is an element $s \in \Lambda(E_4)$, such that $o = ds$.) The set of all exact (closed) forms makes up the subspace $\Upsilon(E_4)(\tilde{\Upsilon}(E_4))$ over space R , but not over $\Lambda^0(E_4)$: $\Upsilon(E_4) \subset \tilde{\Upsilon}(E_4)$ [19]. If S is a domain in E_4 , which can be contracted to a point (S by a smooth map is a star-shaped domain relative to this point called a centre), then, according to the Poincare-lemma [58], we have $\tilde{\Upsilon}(S) \subset \Upsilon(S)$. Therefore, if $ddo = 0$ for $o \in S$, there is an element $s \in S$, such that $o = ds$.

We shall designate through $\Lambda_{p,q}^k(E_4)$ the space of all $(p \times q)$ -matrices whose components are the exterior differential k -forms defined on $BV(E_4)$. Then $\mathbf{K} \in \Lambda_{3,1}^3(E_4)$ is the column matrix of the 3-forms on E_4 , and $\mathbf{K}^* \in \Lambda_{1,3}^3(E_4)$ is the row matrix of the 3-forms.

Let us consider the second subspace into which the space $\Lambda(E_4)$ is subdivided by the operation of external differentiation. Let

$$w = w_{a_1 \dots a_k}(X^b) dX^{a_1} \wedge \dots \wedge dX^{a_k}$$

be an exterior differential form. Following [19], we shall define the linear integral homotopy operator \hat{H} on a certain star-shaped domain S in E_4 with its centre (X^a) relating to the given coordinate covering, understanding by the integral an integration with respect to a measure (note that the use of inexact forms introduces, by means of the homotopy operator, a non-localization in time and space into the continuous theory of topological defects)

$$\hat{H}w = \int_0^1 \lambda^{k-1} \mathfrak{N} \cdot \tilde{w}(\lambda) d\lambda. \tag{A9}$$

In equation (A9) we have designated

$$\mathfrak{N} = (X^a - X_0^a) \partial_a \tag{A10}$$

$$\tilde{w}(\lambda) = w_{a_1 \dots a_k}(X_0^b + \lambda(X^b - X_0^b)) dX^{a_1} \wedge \dots \wedge dX^{a_k}. \tag{A11}$$

The homotopy operator has the following properties [19]:

(1)

$$\hat{H} : \Lambda^k(S) \implies \Lambda^{k-1}(S) \text{ for } k \geq 1 \quad \hat{H}\Lambda^0(S) = 0 \tag{A12}$$

(2)

$$d\hat{H} + \hat{H}d = \hat{I} \quad \text{for } k \geq 1 \tag{A13}$$

$$(\hat{H}df)(X^a) = f(X^a) - f(X_0^a) \quad \text{for } k = 0 \tag{A14}$$

(3)

$$\hat{H}\hat{H}w(X^a) = 0 \quad \hat{H}w(X_0^a) = 0 \tag{A15}$$

(4)

$$\hat{H}d\hat{H} = \hat{H} \quad d\hat{H}d = d \tag{A16}$$

(5)

$$\mathfrak{N} \cdot \hat{H} = 0 \quad \hat{H}\mathfrak{N} \cdot = 0. \tag{A17}$$

According to equation (A13), any form $w \in \Lambda^k(S) (k \geq 1)$ satisfies the equality

$$W = d\hat{H}w + \hat{H}dw.$$

The element $w_E = d\hat{H}w \in \Upsilon(S)$ is the exact part of the form w . (The elements of the space $\Lambda^0(S)$ do not contain an exact part w .) Let us designate

$$w_A = \hat{H}dw = w - w_E.$$

According to equation (A15), $\hat{H}w_A = 0$. Hence, w_A belongs to the kernel of the homotopy operator, designated as $\ker\hat{H} : w_A \in \ker\hat{H}$. The element $w_A \in \Lambda^k(S)$ is called the inexact part of the form w . The set of all inexact differential forms from $\Lambda^k(S)$ is designated as $\Xi^k(S)$, and $\Lambda^0(S)$ is identified with the space $\Xi^0(S)$. The set of all inexact forms on S is designated as $\Xi(S)$ and represents a submodule of the space $\Lambda(S)$ ($\Xi(S) \subseteq \Lambda(S)$ is such that the mapping $\Xi(S) \times \Xi(S) \implies \tilde{\Xi}(S) \subseteq \Xi(S)$ satisfies the conditions of distributivity ($x(u+v) = xu + xv$; $(x+y)u = xu + yu$), associativity ($xy(v) = x(yv)$), and the property of being unitary ($1 \cdot v = v$). Hence, the subspace $\Xi(S)$ is closed relative to the operation of the external product: each set $\Xi^k(S)$ represents a vector space over $\Lambda^0(S)$, such that $\Xi^k(S) \wedge \Xi^m(S) \subseteq \Xi^{k+m}(S)$ and the external product of elements $\ker\hat{H}$ belongs again to $\ker\hat{H}$. If $w \in \Lambda^k(S)$ for $k \geq 1$, then $w \in \ker\hat{H}$ and the first equation from equations (A15) gives

$$w = \hat{H} dw \quad \text{for } \forall w \in \Xi^k, k \geq 1. \quad (\text{A18})$$

Expression (A18) shows that the linear homotopy operator is converse relative to the operator of external differentiation on the submodule $\Xi(S)$.

Let us define the four-dimensional covariant operator of external differentiation by the expression [19]

$$D = dX^a \wedge D_a \quad D_a = \partial_a + \tilde{G}_a. \quad (\text{A19})$$

In equation (A19) D_a is the covariant derivative and \tilde{G}_a is the 1-form of connectedness. If $\tilde{\mathbf{h}}$ is the matrix of the k -forms which are transformed by the rule ${}^*\tilde{\mathbf{h}} = \mathbf{A}\tilde{\mathbf{h}}$, $\mathbf{A} \in G$, then its covariant derivative

$$D\tilde{\mathbf{h}} = d\tilde{\mathbf{h}} + \tilde{\mathbf{G}} \wedge \tilde{\mathbf{h}} \quad (\text{A20})$$

will be transformed by the rule [19]

$${}^*D({}^*\tilde{\mathbf{h}}) = \mathbf{A}(D\tilde{\mathbf{h}}). \quad (\text{A21})$$

If $\tilde{\mathbf{r}}$ is the matrix of the k -forms which are transformed by the rule ${}^*\tilde{\mathbf{r}} = \tilde{\mathbf{r}}\mathbf{A}^{-1}$, $\mathbf{A} \in G$, then its covariant derivative

$$D\tilde{\mathbf{r}} = d\tilde{\mathbf{r}} - (-1)^k \tilde{\mathbf{r}} \wedge \tilde{\mathbf{G}} \quad (\text{A22})$$

will be transformed according to the rule [19]

$${}^*D({}^*\tilde{\mathbf{r}}) = (D\tilde{\mathbf{r}})\mathbf{A}^{-1}. \quad (\text{A23})$$

Finally, if $\tilde{\mathbf{s}}$ is the matrix of the k -forms which are transformed by the rule ${}^*\tilde{\mathbf{s}} = \mathbf{A}\tilde{\mathbf{s}}\mathbf{A}^{-1}$, $\mathbf{A} \in G$, then its covariant derivative

$$D\tilde{\mathbf{s}} = d\tilde{\mathbf{s}} + \tilde{\mathbf{G}} \wedge \tilde{\mathbf{s}} - (-1)^k \tilde{\mathbf{s}} \wedge \tilde{\mathbf{G}} \quad (\text{A24})$$

will be transformed according to the rule [19]

$${}^*D({}^*\tilde{\mathbf{s}}) = \mathbf{A}(D\tilde{\mathbf{s}})\mathbf{A}^{-1}. \quad (\text{A25})$$

Applying the covariant operator of external differentiation to equations (A20), (A22), and (A24), we obtain [19]

$$DD\tilde{\mathbf{h}} = \tilde{\mathbf{T}} \wedge \tilde{\mathbf{h}} \quad (\text{A26})$$

$$DD\tilde{\mathbf{r}} = -\tilde{\mathbf{r}} \wedge \tilde{\mathbf{T}} \quad (\text{A27})$$

$$DD\tilde{\mathbf{s}} = \tilde{\mathbf{T}} \wedge \tilde{\mathbf{s}} - \tilde{\mathbf{s}} \wedge \tilde{\mathbf{T}}. \quad (\text{A28})$$

In equations (A26), (A27), and (A28), $\tilde{\mathbf{T}} = d\tilde{\mathbf{G}} + \tilde{\mathbf{G}} \wedge \tilde{\mathbf{G}}$ is the matrix of the 2-forms of curvature, which is transformed by the second of the rules (44).

References

- [1] Abramova K B, Zlatin N A and Peregood B P 1975 *Sov. Phys.–JETP* **42** 1019
- [2] Bateman G 1979 *MHD-Instability* (Cambridge, MA: MIT)
- [3] Chase W G and Moore H K (eds) 1959 *Exploding Wires* vol 1 (New York: Plenum)
- [4] Chase W G and Moore H K (eds) 1962 *Exploding Wires* vol 2 (New York: Plenum)
- [5] Chase W G and Moore H K (eds) 1964 *Exploding Wires* vol 3 (New York: Plenum)
- [6] Lebedev S V and Savvatimsky A I 1984 *Sov. Phys.–Usp.* **27** 749
- [7] Burtsev V A, Kalinin N V and Luchinsky A V 1990 *Electrical Explosion of Conductors* (Moscow: Energoatomisdat) (in Russian)
- [8] Iskoldsky A M, Volkov N B and Zubareva O V 1996 *Physica* **91D** 182
- [9] Kesaei I G 1968 *Cathode Processes in Electric Arcs* (Moscow: Nauka) (Engl. transl.: SAND 78-6011, NTIS)
- [10] Rotshtein J 1964 *Exploding Wires* vol 3 (New York: Plenum) p 115
- [11] Mesyats G A and Proskurovsky D I 1989 *Pulsed Electrical Discharge in Vacuum* (Berlin: Springer)
- [12] Volkov N B and Iskoldsky A M 1993 *J. Phys. A: Math. Gen.* **26** 6635
Volkov N B and Iskoldsky A M 1993 *J. Phys. A: Math. Gen.* **26** 6647
Volkov N B and Iskoldsky A M 1993 *J. Phys. A: Math. Gen.* **26** 6667
- [13] Baksht R B, Datsko I M and Korostelev A F 1985 *Sov. Phys.–Tech. Phys.* **30** 772
- [14] Volkov N B, Zubarev N M and Iskoldsky A M 1996 *JETP* **82** 228
- [15] Iskoldsky A M, Volkov N B and Zubarev N M 1996 *Phys. Lett.* **217A** 380
- [16] Pranevichyus L I, Bartashyus I Yu and Iglunas V I 1969 *Isv. Vusov. Ser. Phys.* **4** 44 (in Russian)
- [17] Gabovich M and Poritsky V 1981 *Pis. JETP* **33** 320 (in Russian)
- [18] Volterra V 1907 *Ann. de l'Ecole Norm. Sup., Ser. 3* **24** 401
- [19] Kadić A and Edelen D G B 1983 *A Gauge Theory of Dislocations and Disclinations* (Berlin: Springer)
- [20] Lamb H 1932 *Hydrodynamics* (Cambridge: Cambridge University Press)
- [21] Kosevich A M 1981 *Physical Mechanics of Real Crystals* (Kiev: Naukova Dumka) (in Russian)
- [22] Yang C N and Mills R L 1954 *Phys. Rev.* **96** 191
- [23] Patashinsky A Z and Shumilo B I 1985 *JETP* **89** 315 (in Russian)
- [24] Trzęsowski I 1987 *Int. J. Theor. Phys.* **26** 1059
- [25] Sedov L I 1966 *Applied Mechanics (Proc. 11th Int. Congr. Appl. Mech. (Munich, 1964))* (Berlin: Springer) p 9
- [26] Sedov L I 1968 *Irreversible Aspects of Continuum Mechanics and Transfer of Physical Characteristics of Moving Fluids (Proc. IUTAM Symp. (Vienna, 1966))* (New York: Springer) p 346
- [27] Sedov L I and Tsipkin A G 1989 *Foundations of Macroscopic Theories of Gravitation and Electromagnetic Phenomena* (Moscow: Nauka) (in Russian)
- [28] Vol'pert A I and Hudyayev S I 1975 *Calculus in Set of Discontinuous Functions and Equations of Mathematical Physics* (Moscow: Nauka) (in Russian)
- [29] Ryder L H 1985 *Quantum Field Theory* (Cambridge: Cambridge University Press)
- [30] Daniel M and Viallet C M 1980 *Rev. Mod. Phys.* **52** 175
- [31] Doubrovine B, Novikov S and Fomenko A 1982 *Geometrie Contemporaine. Méthodes et Applications* (Moscow: Mir)
- [32] Helgason S 1978 *Differential Geometry, Lie Groups and Symmetric Spaces* (New York: Academic)
- [33] Richtmayer R D 1981 *Principles of Advanced Mathematical Physics* vol 2 (New York: Springer)
- [34] Trudell K 1972 *A First Course in Rational Continuum Mechanics* (Baltimore: The John Hopkins University)
- [35] Sedov L I 1976 *Mechanics of Continuum Media* vol 1–2 (Moscow: Nauka) (in Russian)
- [36] Maugin G A 1988 *Continuum Mechanics of Electromagnetic Solids* (Amsterdam: North-Holland)
- [37] Andreev A F and Pushkarov D I 1985 *JETP* **89** 1883 (in Russian)
- [38] Landau L D and Lifshits E M 1959 *Mechanics of Continuous Media* (Oxford: Pergamon)
- [39] Landau L D and Lifshits E M 1988 *The Classical Theory of Fields* (Moscow: Nauka) (in Russian)
- [40] Cartan E 1945 *Les Systèmes Différentiels Extérieurs et leurs Applications Géométriques. Exposés de Géométrie* vol 14 (Paris: Hermann)
- [41] Finikov S P 1948 *Method of Cartan Exterior Forms with Applications to the Differential Geometry* (Moscow: GTTI) (in Russian)
- [42] Lifshits I M, Asbel M Ya and Kaganov M I 1971 *Electron Theory of Metals* (Moscow: Nauka) (in Russian)
- [43] Landau L D and Lifshits E M 1974 *Quantum Mechanics. Nonrelativistic Theory* (Moscow: Nauka), an exercise to the paragraph 17 (in Russian)
- [44] Landau L D and Lifshits E M 1959 *Statistical Physics* (London: Pergamon)
- [45] Landau L D and Lifshits E M 1982 *Electrodynamics of Continuous Media* (Moscow: Nauka) (in Russian)

- [46] De Groot S R and Suttrop L G 1972 *Foundations of Electrodynamics* (Amsterdam: North-Holland)
- [47] Harrison W A 1966 *Pseudopotentials in the Theory of Metals* (New York: Benjamin)
- [48] Volkov N B and Nemirovsky A Z 1991 *J. Phys. D: Appl. Phys.* **24** 693
- [49] Umezawa H, Matsumoto H and Tashiki H 1982 *Thermo Field Dynamics and Condensed States* (Amsterdam: North-Holland)
- [50] Landau L D 1941 *J. Phys. USSR* **5** 71
- [51] Landau L D 1944 *J. Phys. USSR* **8** 1
- [52] Khalatnikov I M 1971 *The Theory of Superfluidity* (Moscow: Nauka) (in Russian)
- [53] Volovik G E and Dotsenko V S 1980 *JETF* **78** 132 (in Russian)
- [54] Andreev A F and Kagan M Yu 1984 *JETF* **86** 546 (in Russian)
- [55] Dorovsky V N 1989 *Geology Geophys.* **30** 39
- [56] Dorovsky V N and Perepechko Yu V 1992 *J. Appl. Mech. Tech. Phys.* **33** 403
- [57] Blokhin A M and Dorovsky V N 1994 *Problems of the Mathematical Modelling in the Theory of Multivelocity Continuum* (Novosibirsk: United Inst. of Geology, Geophysics and Mineralogy Publishers) (in Russian)
- [58] Shwartz L 1967 *Analyse Mathematique* vol 1–2 (Paris: Hermann)
- [59] Flanders H 1963 *Differential Forms with Applications to the Physical Sciences* (New York: Academic)